COMMENTATIONES MATHEMATICAE

AVCTORE

Moritz von

MAVRICIO DE PRASSE

MATHEM. PROF. ORD. IN VNIVERS. LITER. LIPSIENSI

FASCICVLVS PRIMVS

LIPSIAE

APVD CHRIST. THEOPH. RABENHORST

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I.

ADDITAMENTA
AD
THEORIAM
SERIERVM ARITHMETICARVM
ORDINVM SVPERIORVM

ADDITAMENTA
AD
THEORIAM SERIERUM ARITHMETICARUM
ORDINVM SVPERIORVM.

§. 1.

Pleraeque de ſeriebus arithmeticis quaeſtiones eam ob cauſam habitae ſunt, ut ſimplicioribus et expeditioribus formulis earum ſummae exprimerentur; quae vero ſive ad earum ordinem, ſive ad conſtructiones diverſas, ſive ad alia pertinerent, paullo remiſſiori cura indagata eſſe videntur. Huius negligentiae quamvis plures excogitari poſſint rationes, duae tamen potiſſimum apud analyſtas ſine dubio valuerunt: ſcilicet olim ſive formularum nimium excurrentium multitudo eos ab illis diſquiſitionibus deterruit, ſive negotium minori alacritate ſusceperunt, quod praemia parciora et incertiora proponeret laboris certi et diuturnitate taedioſi. At noſtris temporibus, quibus inventorum praeclariſſimorum cuiusvis generis ingens multitudo in lucem prolata eſt, atque ſignorum copia ſuppetit expeditiſſimorum et ad adeundos explorandosque omnes fere Analyſeos receſſus aptiſſimorum, incommoda calculorum moles facilius vitatur, minusve reformidatur, neque omnis labor recte effugitur, quin ſtatim, vix opera prima tentatus, fructuum provenientium ubertatem oſtendat, praeſertim cum illo labore natura rei longe lateque patentis accuratius cognoſcatur. Talis enim rei perſcrutatio ſaepe inexpectatos habet eventus, ſemperque ſpem praebet certiorem inventi alicuius novi et utilitate conſpicui. His igitur rationibus permoti ordinem et differentias noviſſimas earum ſerierum exploravimus, quae ex multiplicatione terminorum homologorum binarum, ſive plurium ſerierum arithmeticarum oriuntur, et huius inveſtigationis fructum lectoribus obtulimus.

§. 2.

THEOREMA.

Series arithmetica M ordinis (m)^{ti} per primos differentiarum suarum terminos z, a, b, c .... coefficientium binomialium ope expressa et ita disposita, ut singula linea horizontalis singulum seriei terminum efficiat, hanc formam induit:

$$
\begin{array}{lllllll}
M = & z & & & & & \\
& +[z + & {}^{1}\mathfrak{A}a] & & & & \\
& +[z + & {}^{2}\mathfrak{A}a + & {}^{2}\mathfrak{B}b] & & & \\
& +[z + & {}^{3}\mathfrak{A}a + & {}^{3}\mathfrak{B}b + & {}^{3}\mathfrak{C}c] & & \\
& \quad '' & \quad '' & \quad '' & \quad '' & & \\
& \quad '' & \quad '' & \quad '' & \quad '' & & \\
& +[z + & {}^{k}\mathfrak{A}a + & {}^{k}\mathfrak{B}b + & {}^{k}\mathfrak{C}c \; \ldots\ldots & {}^{k}\mathfrak{K}k] & \\
& \quad '' & \quad '' & \quad '' & \quad '' & \quad '' & \\
& \quad '' & \quad '' & \quad '' & \quad '' & \quad '' & \\
& +[z + & {}^{m}\mathfrak{A}a + & {}^{m}\mathfrak{B}b + & {}^{m}\mathfrak{C}c \; \ldots\ldots & {}^{m}\mathfrak{K}k \; \ldots\ldots & {}^{m}\mathfrak{M}m] \\
& +[z + & {}^{m+1}\mathfrak{A}a + & {}^{m+1}\mathfrak{B}b + & {}^{m+1}\mathfrak{C}c \; \ldots\ldots & {}^{m+1}\mathfrak{K}k \; \ldots\ldots & {}^{m+1}\mathfrak{M}m] \\
& \quad '' & \quad '' & \quad '' & \quad '' & \quad '' & \quad ''
\end{array}
$$

DEMONSTRATIO.

Primarum differentiarum derivatio primam seriem verticalem tollit, secundarum secundam, (k)^{tarum} (k)^{tam}. Namque differentiae coefficientium binomialium seriei verticalis (k+1)^{tae} ad coefficientes binomialis seriei verticalis (k)^{tae} redeunt, quippe ${}^{m+1}\mathfrak{K}k - {}^{m}\mathfrak{K}k = {}^{m-1}\overset{-1}{\mathfrak{K}}k$, cumque sint series verticales numero (m+1), non nisi derivatione differentiarum (m+1)^{tarum} ultima series ad dextram evanescit ipsarumque differentiarum series m deprehenduntur, unde sequitur, ut series M sit ordinis (m)^{ti}.

COROLLARIVM. Ex formula serierum arithmeticarum proposita simul patet, easdem esse differentias (m)^{tas} ipsius seriei M et terminorum novissimae (m+1)^{tae} columnae, seu seriei verticalis, quae terminos continet ${}^{m}\mathfrak{M}m$; ${}^{m+1}\mathfrak{M}m$; ${}^{m+2}\mathfrak{M}m$; ..... Namque (m)^{tis} differentiis derivatis, de (m+1)^{ta} columna supersunt differentiae novissimae, reliqua omnia evanuerunt.

SCHOLION I. Adhibitae sunt notae HINDENBURGIANAE coefficientium binomialium, iam ita pervulgatae, ut paene supervacuum arbitratus sim

ſim apponere ${}^{m}\mathfrak{A} = \frac{m}{1}$; ${}^{m}\mathfrak{B} = \frac{m(m-1)}{1.2}$; .... $\overset{-1}{{}^{m}\mathfrak{K}} = \frac{m(m-1)\ldots(m-k+2)}{1.2\ldots.(k-1)}$

$${}^{m}\mathfrak{K} = \frac{m(m-1)\ldots.(m-k+1)}{1.2\ldots\ldots.k} \ldots.$$

SCHOLION II. Alii aliis formulis ſeries arithmeticas expreſſerunt, v. c. KAESTNERUS (*Anfangsgründe der Analyſis endlicher Gröſsen* §. 724. pag. 504.).

§. 3.
EXPLICATIO.

In formula ſeriei *M* §. 2. propoſita ſingulae ſeries verticales, ſive columnae, ſingulas arithmeticas ſeries conſtituunt; ſic prima ſeriem ordinis (o), ſecunda primi, tertia ſecundi ordinis ſeriem exhibet etc. Praeterea haec formula generalis *M* ad ſeries arithmeticas quoque eas, quas ſingulae columnae exhibent, transferri poteſt:

e. g. ${}^{2}\mathfrak{B}b = b$
$+{}^{3}\mathfrak{B}b = b + {}^{1}\mathfrak{A}2b$
$+{}^{4}\mathfrak{B} = b + {}^{2}\mathfrak{A}2b + {}^{2}\mathfrak{B}b$
$+{}^{5}\mathfrak{B} = b + {}^{3}\mathfrak{A}2b + {}^{3}\mathfrak{B}b$
" " " "
" " " "

nec tamen vice verſa ſeries arithmetica quaelibet unam in columnam formulae *M* §. 2. ſe cogi patitur, quod haec reductio peculiarem quantitatum *z*, *a*, *b*, *c* .... rationem requirit.

Ita ſeries z
$+(z + {}^{1}\mathfrak{A}a)$
$+(z + {}^{2}\mathfrak{B}a + {}^{2}\mathfrak{B}b)$
$+(z + {}^{3}\mathfrak{B}a + {}^{3}\mathfrak{B}b)$
" " "
" " "

nunquam ad ſeriem formae ${}^{2}\mathfrak{B}x$; ${}^{3}\mathfrak{B}x$; ${}^{4}\mathfrak{B}x$; ${}^{5}\mathfrak{B}x$; . . . redibit, quicunque tribuatur valor literae *x*, niſi ſit $z = \frac{1}{2}a = b = x$ id quod ſuperiori exemplo patet.

Illae ſeries, quae ad ſingulam columnam formulae *M* revocari poſſunt, *ſeries arithmeticae ſimplices*, reliquae vero *compoſitae* vocabuntur, huiusque noſtrae diſtinctionis cauſa et utilitas infra patebit.

§. 4.
THEOREMA.

Summae terminorum homologorum duarum ſerierum arithmeticarum, quarum altera inferioris ordinis $(p)^{d}$ eſt, altera ſuperioris $(q)^{ti}$, ſeriem conſtituunt arithmeticam ordinis ſuperioris $(q)^{ti}$.

DEMON-

DEMONSTRATIO.

Vt numeris certis utamur et calculis parcamus, ſit e. g. $p = 2$; $q = 4$; erit §. 2. forma:

prioris ſeriei:	poſterioris:	ſummarum terminorum homologorum:
x	y	$(x+y)$
$+(x+{}^{1}\mathfrak{A}a)$	$+(y+{}^{1}\mathfrak{A}\alpha)$	$+[(x+y)+{}^{1}\mathfrak{A}(a+\alpha)]$
$+(x+{}^{2}\mathfrak{A}a+{}^{2}\mathfrak{B}b)$	$+(y+{}^{2}\mathfrak{A}\alpha+{}^{2}\mathfrak{B}\beta)$	$+[(x+y)+{}^{2}\mathfrak{A}(a+\alpha)+{}^{2}\mathfrak{B}(b+\beta)]$
$+(x+{}^{3}\mathfrak{A}a+{}^{3}\mathfrak{B}b)$	$+(y+{}^{3}\mathfrak{A}\alpha+{}^{3}\mathfrak{B}\beta+{}^{3}\mathfrak{C}\gamma)$	$+[(x+y)+{}^{3}\mathfrak{A}(a+\alpha)+{}^{3}\mathfrak{B}(b+\beta)+{}^{3}\mathfrak{C}\gamma]$
$+(x+{}^{4}\mathfrak{A}a+{}^{4}\mathfrak{B}b)$	$+(y+{}^{4}\mathfrak{A}\alpha+{}^{4}\mathfrak{B}\beta+{}^{4}\mathfrak{C}\gamma+{}^{4}\mathfrak{D}\delta)$	$+[(x+y)+{}^{4}\mathfrak{A}(a+\alpha)+{}^{4}\mathfrak{B}(b+\beta)+{}^{4}\mathfrak{C}\gamma+{}^{4}\mathfrak{D}\delta]$
$+(x+{}^{5}\mathfrak{A}a+{}^{5}\mathfrak{B}b)$	$+(y+{}^{5}\mathfrak{A}\alpha+{}^{5}\mathfrak{B}\beta+{}^{5}\mathfrak{C}\gamma+{}^{5}\mathfrak{D}\delta)$	$+[(x+y)+{}^{5}\mathfrak{A}(a+\alpha)+{}^{5}\mathfrak{B}(b+\beta)+{}^{5}\mathfrak{C}\gamma+{}^{5}\mathfrak{D}\delta]$
" " "	" " " " "	" " " " "

Summae igitur terminorum homologorum ad formam ſeriei *M* §. 2. propoſitae redeunt et ordinis quarti ſeriem efficiunt. Eadem ratiocinia cum ad quascunque ſeries arithmeticas poſſint transferri, patet, exemplum allatum demonſtrationis vicem ſuſtinere.

COROLLARIUM.

Series ſummarum terminorum homologorum cum ſerie ordinis ſuperioris communes habet differentias noviſſimas, quarum valor in exemplo propoſito quidem $= \delta$. Praeterea theorema ad plures, quam duas, ſeries extendi poteſt.

§. 5.

THEOREMA.

Eſt $A = {}^{p}\mathfrak{N}x + {}^{p+1}\mathfrak{N}x + {}^{p+2}\mathfrak{N}x$ .... ſeries arithmetica ſimplex ordinis $(n)^{ti}$.

$B = y + (y+z) + (y+2z)$ .... ſeries arithmetica quaelibet ordinis primi, terminis utriusque ſeriei homologis multiplicatione iunctis, proveniet

$C = {}^{p}\mathfrak{N}xy + {}^{p+1}\mathfrak{N}x(y+z) + {}^{p+2}\mathfrak{N}x(y+2z)$ .....

eritque C ſeries arithmetica ordinis $(n+1)^{ti}$ eiusque differentiae noviſſimae erunt $= (n+1)\, xz$.

DEMONSTRATIO.

Producta C inſtruantur in ſerie verticali, ad dextram adponantur differentiae terminorum primae, ſecundae, tertiae, prodibunt columnae ſequentes:

${}^{p}\mathfrak{N}xy$

	Diff. I^{mae}	Diff. II^{dae}	Diff. III^{tiae}
${}^{p}\mathfrak{N}xy$			
${}^{p+1}\mathfrak{N}x(y+z)$	${}^{p+1}\mathfrak{N}xz+{}^{p}\overset{-1}{\mathfrak{N}}xy$		
${}^{p+2}\mathfrak{N}x(y+2z)$	${}^{p+2}\mathfrak{N}xz+{}^{p+1}\overset{-1}{\mathfrak{N}}x(y+z)$	$2\,{}^{p+1}\overset{-1}{\mathfrak{N}}xz+{}^{p}\overset{-2}{\mathfrak{N}}xy$	
${}^{p+3}\mathfrak{N}x(y+3z)$	${}^{p+3}\mathfrak{N}xz+{}^{p+2}\overset{-1}{\mathfrak{N}}x(y+2z)$	$2\,{}^{p+2}\overset{-1}{\mathfrak{N}}xz+{}^{p+1}\overset{-2}{\mathfrak{N}}x(y+z)$	$3\,{}^{p+1}\overset{-2}{\mathfrak{N}}xz+{}^{p}\overset{-3}{\mathfrak{N}}xy$
${}^{p+4}\mathfrak{N}x(y+4z)$	${}^{p+4}\mathfrak{N}xz+{}^{p+3}\overset{-1}{\mathfrak{N}}x(y+3z)$	$2\,{}^{p+3}\overset{-1}{\mathfrak{N}}xz+{}^{p+2}\overset{-2}{\mathfrak{N}}x(y+2z)$	$3\,{}^{p+2}\overset{-2}{\mathfrak{N}}xz+{}^{p+1}\overset{-3}{\mathfrak{N}}x(y+z)$
"	" "	" "	" "
"	" "	" "	" "

His praemissis, eandem legem et reliquas differentias servare fingamus, eruntque differentiae $(k)^{tae}$

	e quibus sequentur differentiae $(k+1)^{tae}$
$k\,{}^{p+1}\overset{-k+1}{\mathfrak{N}}xz+{}^{p}\overset{-k}{\mathfrak{N}}xy$	$(k+1)\,{}^{p+1}\overset{-k}{\mathfrak{N}}xz+{}^{p}\overset{-k-1}{\mathfrak{N}}xy$
$k\,{}^{p+2}\overset{-k+1}{\mathfrak{N}}xz+{}^{p+1}\overset{-k}{\mathfrak{N}}x(y+z)$	$(k+1)\,{}^{p+2}\overset{-k}{\mathfrak{N}}xz+{}^{p+1}\overset{-k-1}{\mathfrak{N}}x(y+z)$
$k\,{}^{p+3}\overset{-k+1}{\mathfrak{N}}xz+{}^{p+2}\overset{-k}{\mathfrak{N}}x(y+2z)$	$(k+1)\,{}^{p+3}\overset{-k}{\mathfrak{N}}xz+{}^{p+2}\overset{-k-1}{\mathfrak{N}}x(y+2z)$
$k\,{}^{p+4}\overset{-k+1}{\mathfrak{N}}xz+{}^{p+3}\overset{-k}{\mathfrak{N}}x(y+3z)$	
" "	" "
" "	" "

Eadem igitur differentiarum lex obtinet non modo pro $k = 1, 2, 3$, sed etiam pro maioribus literae k valoribus, quippe differentiae $(k+1)^{tae}$, a differentiis $(k)^{tis}$ derivatae, ab illa lege non recedunt. E coefficientium binomialium vero natura intelligitur, valores literae k qui numerum (n) superant, quod coefficientes hic occurrentes pro maioribus illis valoribus evanescunt, hoc loco excludi, maximumque omnium valorum ipsum esse numerum (n), quo introducto nanciscimur differentias $(n+1)^{tas}$ novissimas

$$(n+1)\,{}^{p+1}\overset{-n}{\mathfrak{N}}xz+{}^{p}\overset{-n-1}{\mathfrak{N}}xy = (n+1)xz$$
$$(n+1)\,{}^{p+2}\overset{-n}{\mathfrak{N}}xz+{}^{p+1}\overset{-n-1}{\mathfrak{N}}x(y+z) = (n+1)xz$$
" " " "

quae simul aequalitate sua ostendunt, seriem productorum esse ordinis $(n+1)^{ti}$.

§. 6.

THEOREMA.

Terminis homologis serierum arithmeticarum simplicium (§. 2.),

${}^{n}\mathfrak{S}p$; ${}^{n+1}\mathfrak{S}p$; ${}^{n+2}\mathfrak{S}p$; ${}^{n+3}\mathfrak{S}p$ . . . . . (P

${}^{m}\mathfrak{K}q$; ${}^{m+1}\mathfrak{K}q$; ${}^{m+2}\mathfrak{K}q$; ${}^{m+3}\mathfrak{K}q$ . . . . . (Q

quarum

quarum altera P est ordinis $(s)^{ti}$, altera Q ordinis $(k)^{ti}$, multiplicatione iunctis exoritur ${}^{n}\mathfrak{S}p^{m}\mathfrak{K}q$; ${}^{n+1}\mathfrak{S}p^{m+1}\mathfrak{K}q$; ${}^{n+2}\mathfrak{S}p^{m+2}\mathfrak{K}q$; ${}^{n+3}\mathfrak{S}p^{m+3}\mathfrak{K}q$; .... series ordinis $(k+s)^{ti}$, cuius differentiae novissimae erunt $= {}^{s+k}\mathfrak{K}pq = {}^{s+k}\mathfrak{S}pq$.

DEMONSTRATIO.

Secundum §. 5.

$$ {}^{n}\mathfrak{S}\frac{m}{1};\ {}^{n+2}\mathfrak{S}\frac{m+1}{1};\ {}^{n+1}\mathfrak{S}\frac{m+1}{1} \ldots\ldots = {}^{n}\mathfrak{S}^{m}\mathfrak{A};\ {}^{n+1}\mathfrak{S}^{n+1}\mathfrak{A};\ {}^{n+2}\mathfrak{S}^{n+2}\mathfrak{A};\ \ldots\ldots $$

series est ordinis $(s+1)^{ti}$, cuius differentiae novissimae sunt $\frac{s+1}{1} = {}^{s+1}\mathfrak{A}$.

Similiter

$$ {}^{n}\mathfrak{S}^{m}\mathfrak{A}\frac{m-1}{2};\ {}^{n+1}\mathfrak{S}^{m+1}\mathfrak{A}\frac{m}{2};\ {}^{n+2}\mathfrak{S}^{m+2}\mathfrak{A}\frac{m+1}{2};\ldots = {}^{n}\mathfrak{S}^{m}\mathfrak{B};\ {}^{n+1}\mathfrak{S}^{m+1}\mathfrak{B};\ {}^{n+2}\mathfrak{S}^{m+2}\mathfrak{B};\ldots $$

est series ordinis $(s+2)^{ti}$ cuius differentiae novissimae sunt $(s+2)\,{}^{s+1}\mathfrak{A}\frac{1}{2} = {}^{s+2}\mathfrak{B}$.

His igitur ratiociniis, quae nobis pro lubitu continuare licet, perveniri tandem intelligitur ad seriem, quae, si terminis omnibus adiiciantur factores pq, propositam seriem productorum referat, ad ordinem $(s+k)^{tum}$ adscendat et differentias habeat novissimas $= \frac{(s+k).(s+k-1)\ \ldots\ (s+1)}{1\ .\ 2\ \ldots\ldots\ k}\ pq$ $= {}^{s+k}\mathfrak{K}pq = {}^{s+k}\mathfrak{S}pq$.

§. 7.

THEOREMA.

Sint P et Q series arithmeticae quaelibet, altera quidem P ordinis $(p)^{ti}$, altera R ordinis $(r)^{ti}$, eaeque ita exprimantur:

$$ P = {}^{1}P + {}^{2}P + {}^{3}P + {}^{4}P \ldots\ldots, $$

$$ R = {}^{1}R + {}^{2}R + {}^{3}R + {}^{4}R \ldots\ldots, $$

Series productorum ${}^{1}P^{1}R + {}^{2}P^{2}R + {}^{3}P^{3}R + {}^{4}P^{4}R \ldots\ldots$, ex terminorum homologorum multiplicatione provenientium, ordinis erit $(p+r)^{ti}$, differentias habebit novissimas $= {}^{p+r}\mathfrak{P}p\varrho = {}^{p+r}\mathfrak{R}p\varrho$, ubi p et ϱ novissimas significant differentias serierum P, R.

DEMONSTRATIO.

I. Quia P et Q sunt ordinis $(p)^{ti}$ et $(r)^{ti}$, secundum §. 2. eas nobis ita exprimere licebit: . . . .

${}^{1}P = x$

$$P=\begin{cases} {}^{1}P= x \\ +{}^{2}P=+[x + {}^{1}\mathfrak{A}a] \\ +{}^{3}P=+[x + {}^{2}\mathfrak{A}a + {}^{2}\mathfrak{B}b] \\ \quad '' \quad '' \quad '' \quad '' \\ \quad '' \quad '' \quad '' \quad '' \\ +{}^{p+1}P=+[x + {}^{p}\mathfrak{A}a + {}^{p}\mathfrak{B}b \,.. + {}^{p}\mathfrak{P}p] \\ \quad '' \quad '' \quad '' \quad '' \quad '' \\ \quad '' \quad '' \quad '' \quad '' \quad '' \\ +{}^{r+1}P=+[x + {}^{r}\mathfrak{A}a + {}^{r}\mathfrak{B}b \,.. + {}^{r}\mathfrak{P}p] \\ \quad '' \quad '' \quad '' \quad '' \quad '' \\ \quad '' \quad '' \quad '' \quad '' \quad '' \\ +{}^{q+1}P=+[x + {}^{q}\mathfrak{A}a + {}^{q}\mathfrak{B}b \,.. + {}^{q}\mathfrak{P}p] \\ +{}^{q+2}P=+[x + {}^{q+1}\mathfrak{A}a + {}^{q+1}\mathfrak{B}b \,.. + {}^{q+1}\mathfrak{P}p] \\ \quad '' \quad '' \quad '' \quad '' \quad '' \\ \quad '' \quad '' \quad '' \quad '' \quad '' \end{cases} \qquad R=\begin{cases} {}^{1}R= y \\ +{}^{2}R=+[y + {}^{1}\mathfrak{A}\alpha] \\ +{}^{3}R=+[y + {}^{2}\mathfrak{A}\alpha + {}^{2}\mathfrak{B}\beta] \\ \quad '' \quad '' \quad '' \quad '' \\ \quad '' \quad '' \quad '' \quad '' \\ +{}^{p+1}R=+[y + {}^{p}\mathfrak{A}\alpha + {}^{p}\mathfrak{B}\beta \,.. + {}^{p}\mathfrak{P}\pi] \\ \quad '' \quad '' \quad '' \quad '' \quad '' \\ \quad '' \quad '' \quad '' \quad '' \quad '' \\ +{}^{r+1}R=+[y + {}^{r}\mathfrak{A}\alpha + {}^{r}\mathfrak{B}\beta \,.. + {}^{r}\mathfrak{P}\pi \,.. + {}^{r}\mathfrak{R}\varrho] \\ \quad '' \quad '' \quad '' \quad '' \quad '' \quad '' \\ \quad '' \quad '' \quad '' \quad '' \quad '' \quad '' \\ +{}^{q+1}R=+[y + {}^{q}\mathfrak{A}\alpha + {}^{q}\mathfrak{B}\beta \,.. + {}^{q}\mathfrak{P}\pi \,.. + {}^{q}\mathfrak{R}\varrho] \\ +{}^{q+2}R=+[y + {}^{q+1}\mathfrak{A}\alpha + {}^{q+1}\mathfrak{B}\beta \,.. + {}^{q+1}\mathfrak{P}\pi \,.. + {}^{q+1}\mathfrak{R}\varrho] \\ \quad '' \quad '' \quad '' \quad '' \quad '' \quad '' \\ \quad '' \quad '' \quad '' \quad '' \quad '' \quad '' \end{cases}$$

II. Terminis homologis serierum P et Q multiplicatione iunctis, termini generales ${}^{q+1}P$; ${}^{q+2}P$; ${}^{q+3}P$; . . . seriei P ducti in respondentes, terminos generales ${}^{q+1}R$; ${}^{q+2}R$; ${}^{q+3}R$; . . . seriei R haec gignent producta:

$$\begin{aligned} {}^{q+1}P\,{}^{q+1}R &= [x + {}^{q}\mathfrak{A}a + {}^{q}\mathfrak{B}b \ldots + {}^{q}\mathfrak{P}p].[y + {}^{q}\mathfrak{A}\alpha + {}^{q}\mathfrak{B}\beta \ldots + {}^{q}\mathfrak{R}\varrho] \\ {}^{q+2}P\,{}^{q+2}R &= [x + {}^{q+1}\mathfrak{A}a + {}^{q+1}\mathfrak{B}b \ldots + {}^{q+1}\mathfrak{P}p].[y + {}^{q+1}\mathfrak{A}\alpha + {}^{q+1}\mathfrak{B}\beta \ldots + {}^{q+1}\mathfrak{R}\varrho] \\ {}^{q+3}P\,{}^{q+3}R &= [x + {}^{q+2}\mathfrak{A}a + {}^{q+2}\mathfrak{B}b \ldots + {}^{q+2}\mathfrak{P}p].[y + {}^{q+2}\mathfrak{A}\alpha + {}^{q+2}\mathfrak{B}\beta \ldots + {}^{q+2}\mathfrak{R}\varrho] \\ '' \quad '' \quad & \quad '' \quad '' \quad '' \quad\quad '' \quad\quad '' \quad '' \quad '' \\ '' \quad '' \quad & \quad '' \quad '' \quad '' \quad\quad '' \quad\quad '' \quad '' \quad '' \end{aligned}$$

Multiplicatione, quae hic indicatur, revera peracta, terminis primi producti in linea horizontali dispositis, et terminis reliquorum productorum rite subiectis, prodibunt columnae numero $(1+p)\,(1+r)$, quae singulae singulas efficiunt series arithmeticas ordinum diversorum, summi vero ordinis series erit columna ex terminis factorum novissimis conflata, cuius termini spatio parcendi causa in linea horizontali positi, sunt sequentia producta:

$${}^{q}\mathfrak{P}p\;{}^{q}\mathfrak{R}\varrho;\quad {}^{q+1}\mathfrak{P}p\;{}^{q+1}\mathfrak{R}\varrho;\quad {}^{q+2}\mathfrak{P}p\;{}^{q+2}\mathfrak{R}\varrho;\quad {}^{q+3}\mathfrak{P}p\;{}^{q+3}\mathfrak{R}\varrho;\ \ldots\ldots$$

quae seriem arithmeticam ordinis $(p+r)^{ti}$ efficiunt.

III. Secundum §. 4. Corollarium series productorum

$${}^{1}P\,{}^{1}R;\quad {}^{2}P\,{}^{2}R;\quad {}^{3}P\,{}^{3}R;\quad {}^{4}P\,{}^{4}R;\ \ldots\ldots\ldots$$

et series modo proposita

$${}^{q}\mathfrak{P}p\;{}^{q}\mathfrak{R}\varrho;\quad {}^{q+1}\mathfrak{P}p\;{}^{q+1}\mathfrak{R}\varrho;\quad {}^{q+2}\mathfrak{P}p\;{}^{q+2}\mathfrak{R}\varrho;\quad {}^{q+3}\mathfrak{P}p\;{}^{q+3}\mathfrak{R}\varrho;\ \ldots\ldots$$

quae

quae constat terminis ultimae columnae, easdem habent novissimas differentias, quarum valor §. 6. invenitur $= {}^{q+r}\mathfrak{P}pq = {}^{q+r}\mathfrak{R}pq$.

COROLLARIUM.

Ex propositis sequitur, ut, si per multiplicationem termini homologi plurium serierum iungantur, ea re exoriantur series productorum, quarum et ordo et novissimarum differentiarum valor facillime possunt indicari. Sint e. g. series:

$$\left.\begin{array}{llllll} {}^1P; & {}^2P; & {}^3P; & {}^4P; & {}^5P; & \ldots \\ {}^1Q; & {}^2Q; & {}^3Q; & {}^4P; & {}^5Q; & \ldots \\ {}^1R; & {}^2R; & {}^3R; & {}^4P; & {}^5R; & \ldots \\ {}^1S; & {}^2S; & {}^3S; & {}^4S; & {}^5S; & \ldots \\ '' & '' & '' & '' & '' & \\ '' & '' & '' & '' & '' & \end{array}\right\} \begin{array}{c}\text{ordinis } p^{ti},\ q^{ti},\ r^{ti},\ s^{ti} \ldots \\ \text{et novissimas differentias habeant,} \\ P,\ Q,\ R,\ S \ldots\end{array}$$

tum series horizontales:

$$\begin{array}{llllll} \text{I.}\ {}^1P\,{}^1Q; & {}^2P\,{}^2Q; & {}^3P\,{}^3Q; & {}^4P\,{}^4Q; & {}^5P\,{}^5Q; & \ldots \\ {}^1P\,{}^1Q\,{}^1R; & {}^2P\,{}^2Q\,{}^2R; & {}^3P\,{}^3Q\,{}^3R; & {}^4P\,{}^4Q\,{}^4R; & {}^5P\,{}^5Q\,{}^5R; & \ldots \\ {}^1P\,{}^1Q\,{}^1R\,{}^1S; & {}^2P\,{}^2Q\,{}^2R\,{}^2S; & {}^3P\,{}^3Q\,{}^3R\,{}^3S; & {}^4P\,{}^4Q\,{}^4R\,{}^4S; & {}^5P\,{}^5Q\,{}^5R\,{}^5S; & \ldots \\ '' & '' & '' & '' & '' & \\ '' & '' & '' & '' & '' & \end{array}$$

$$\text{erunt ordinis}\ \left.\begin{array}{l}(p+q)^{ti} \\ (p+q+r)^{ti} \\ (p+q+r+s)^{ti} \\ ''\quad '' \\ ''\quad ''\end{array}\right\} \begin{array}{c}\text{novissimas differentias} \\ \text{habebunt}\end{array} \left\{\begin{array}{l} {}^{p+q}\mathfrak{Q}PQ \\ {}^{p+q+r}\mathfrak{R}\ {}^{p+q}\mathfrak{Q}PQR \\ {}^{p+q+r+s}\mathfrak{S}\,{}^{p+q+r}\mathfrak{R}\,{}^{p+q}\mathfrak{Q}PQRS \\ ''\quad ''\quad '' \\ ''\quad ''\quad '' \end{array}\right.$$

positoque ${}^mP = {}^mQ = {}^mR$; series horizontales:

$$\left.\begin{array}{lllll} \text{II.}\ {}^1P^2; & {}^2P^2; & {}^3P^2; & {}^4P^2; & \ldots \\ {}^1P^3; & {}^2P^3; & {}^3P^3; & {}^4P^3; & \ldots \\ '' & '' & '' & '' & \\ '' & '' & '' & '' & \\ {}^1P^n; & {}^2P^n; & {}^3P^n; & {}^4P^n; & \ldots \end{array}\right\} \begin{array}{c}\text{erunt} \\ \text{ordinis}\end{array} \left\{\begin{array}{l}(2p)^{ti} \\ (3p)^{ti} \\ '' \\ '' \\ (np)^{ti}\end{array}\right. \begin{array}{c}\text{differentias} \\ \text{novissimas} \\ \text{habebunt}\end{array} \left\{\begin{array}{l} {}^{2p}\mathfrak{P}P^2 \\ {}^{3p}\mathfrak{P}\,{}^{2p}\mathfrak{P}P^3 \\ ''\quad '' \\ ''\quad '' \\ {}^{np}\mathfrak{P}\,{}^{(n-1)p}\mathfrak{P} \ldots {}^{3p}\mathfrak{P}\,{}^{2p}\mathfrak{P}P^n \end{array}\right.$$

EXEMPLUM I. Series 5, 8, 12, 19, 31, 50, 78, 117 . . . . est ordinis tertii eiusque novissimae differentiae sunt = 2

et series 1, 5, 12, 22, 35, 51, 70, 92 . . . . est ordinis secundi eiusque differentiae novissimae sunt = 3. Hinc sequitur ex corollarii formulis, ut series productorum:

1.5, 5.8, 12.12, 22.19, 35.25, 51.50, 70.78, 92.117 . . . .
5, 40, 144, 418, 1085, 2550, 5460, 10764 . . . .
sit ordinis quinti et differentias habeat novissimas ${}^{3+2}\mathfrak{B}.\ 2.3. = 60$, in formulis enim Corollarii I^{mis} valores literarum p, q, $\mathfrak{Q}$, $\varkappa$, ϱ.
h. l. sunt 3, 2, $\mathfrak{B}$, 2, 3.

EXEMPLUM II. Series 3, 8, 14, 23, 37, 58, 88, 129, 183 . . . est tertii ordinis et differentias novissimas habet = 2.
Series 4, 6, 9, 13, 18, 24, 31, 39, 48 . . . est secundi ordinis et differentias novissimas habet = 1.
Series 3, 7, 11, 15, 19, 23, 27, 31, 35 . . . est ordinis primi et differentias habet omnes = 4. Itaque ex exempli I. rationibus series productorum:

3.4.3, 8.6.7, 14.9.11, 23.13.15, 37.18.19, 58.24.23, 88.31.27, 129.39.31, 183.48.35 . . .
h. e. 36, 336, 1386, 4485, 12654, 32016, 73656, 155961, 307440 . . . est ordinis $(3+2+1)^{ti} = 6^{ti}$ et differentias novissimas habet ${}^{3+2+1}\mathfrak{X}\,{}^{3+2}\mathfrak{B}\, 2.1.4 = 6.10.8 = 480$, literarum enim in formulis I^{mis} Corollarii adhibitarum p, q, r, $\mathfrak{R}$, $\mathfrak{Q}$, $\varkappa$, ϱ, $\imath$
h. l. sunt valores 3, 2, 1, $\mathfrak{A}$, $\mathfrak{B}$, 2, 1, 4

EXEMPLUM III. Series 1^2, 5^2, 15^2, 35^2, 70^2, 126^2, 210^2, 330^2, 495^2, 715^2, 1001^2 . . . . est ordinis octavi et differentias novissimas h. e. octavas habet $= {}^{2.4}\mathfrak{D}1^2 = \frac{8.7.6.5}{1.2.3.4} = 70$, in formulis scilicet Corollarii II^{dis} valores literarum p, $\mathfrak{P}$, $\varkappa$.
h. l. sunt 4, $\mathfrak{D}$, 1.

EXEMPLUM IV. Series 1^m, 2^m, 3^m, 4^m, 5^m, ... est ordinis $(m)^{ti}$ et differentiarum valor communis est m, (m-1) . . . . 3. 2, nam in formulis II^{dis} Corollarii valores literarum p, $\mathfrak{P}$, $\varkappa$, a.
h. l. sunt 1, $\mathfrak{A}$, 1, m.

EXEMPLUM V.

$$\text{Series}\begin{cases}1^2, 2^2, 3^2, 4^2 \ldots\\ 1^3, 2^3, 3^3, 4^3 \ldots\\ 1^5, 2^5, 3^5, 4^5 \ldots\end{cases}\text{sunt ordinis}\begin{cases}2 \text{ differentias}\\ 3 \text{ novissimas}\\ 5 \quad \text{habent}\end{cases}\begin{cases}1.2\\ 1.2.3\\ 1.2.3.4.5\end{cases}$$

ideoque series 1^{10}, 2^{10}, 3^{10}, 4^{10}, 5^{10} . . . . . est ordinis $(2+3+5)^{mi} = 10^{mi}$ et differentias habet novissimas ${}^{2+3+5}\mathfrak{C}\ {}^{2+3}\mathfrak{C}\ (1.2.)\ (1.2.3.)\ (1.2.3.4.5.)$ $= \frac{10.9.8.7.6}{1.2.3.4.5} \cdot \frac{5.4.3}{1.2.3} \cdot (12) \cdot (1.2.3) \cdot (13.3.4.5.) = 10.9.8.7.6.5.4.3.2.$ secundum formu-

formulas Corollarii I^{mae} positis pro p, q, r, $\mathfrak{Q}$, $\mathfrak{R}$, $\mathfrak{r}$, ρ, π.
valoribus 2, 3, 5, $\mathfrak{C}$, $\mathfrak{E}$, 1.2, $1.2.3$, $1.2.4.5$.
Idem potuisset, uti in Exemplo III., reperiri ex formulis II^{dis} Corollarii.

SCHOLION. Seriei 1^m, 2^m, 3^m . . . . ordinem et novissimas differentias KAESTNERVS methodo investigavit alia, videantur: *Anfangsgründe der Analysis endlicher Grössen*, p. 191. §. 310. quocum loco conferantur ea, quae p. 31. §. 47. protulit.

Maior quidem exemplorum numerus, quae varietate sua forte non displicuissent, adiici, nec non, ut novissimarum, sic priorum quoque differentiarum cognitio institui potuisset, qua sine dubio nonnulla a materia huius libelli non aliena allata forent. Sed utrumque omisimus, et hic figendum esse pedem duximus, quia haec conscripta sunt, ut fundamento veluti essent interpolationi serierum arithmeticarum sequenti. Cum scilicet ante aliquot annos sese nobis obtulisset occasio diversa interpolationum genera diligentius explorandi, in res nonnullas incidimus, quae novitatis faciem haberent, et earum rationes dum conquirebantur, libelli huius theoremata congesta sunt. Arctissimo igitur vinculo haec et posterior commentatio cohaerent, diversae tamen naturae argumenta gemini libelli complectuntur; ea enim, quae modo protulimus, universae theoriae arithmeticarum serierum particulam exiguam efficiunt et alioquin varium usum praestitura esse videntur, quae vero in sequenti libello continentur, maximam partem unice ad interpolationem tendunt.

II.

METHODVS NOVA SERIES ARITHMETICAS INTERPOLANDI.

METHODUS NOVA
SERIES ARITHMETICAS INTERPOLANDI.

§. 1.

Neminem dubitare arbitramur, quin praestet, plures nosse vias, ad eundem finem quae perducant, quam aliarum nescium, in eadem semper exerceri. Quaelibet enim e diversis negotii expediendi rationibus commoda praebet peculiaria, quae a reliquis frustra expectentur, et sive hanc, sive illam pro rei peragendae natura praeferri iubeant. Itaque non nullius utilitatis videtur ostendere, duplici modo series arithmeticas cuiuslibet ordinis posse interpolari, praesertim quod hoc negotio saepissime implicamur, quando ad tabulas construendas in solutionibus problematum indirectis confugiendum est. Altera quidem harum methodorum omnibus nota per indices terminorum fractos peragitur, (v. Euleri *Institutt. Calc. diff.* cap. 17.) alteram vero a nemine traditam reperimus. Hactenus scilicet, cum arithmetica series erat interpolanda, eius terminis termini aliquot novi interiiciebantur, ita quidem comparati, ut termini dati cum interpositis seriem efficerent, quacunque lege procedentem; caeterum et legis huius et ipsius seriei interpolatae natura nisi interpolatores latuit, illi certe, quos adivimus, de hac re nihil commemorarunt. Ex *Additamentis* nostris *ad theoriam serierum arithmeticarum* facile ostendi potest, terminos serierum arithmeticarum pariter series constituere arithmeticas, unde colligitur, seriem arithmeticarum interpolare idem esse, ac fingere, seriem datam constare terminis aequidistantibus alterius seriei arithmeticae, huiusque terminos interiacentes restituere; et haec quidem est altera interpolandarum serierum arithmeticarum methodus, quam novam arbitrati hic proponemus.

§. 2.

§. 2.

THEOREMA.

Est summa terminorum:

$$ {}^{n}\mathfrak{M}\,{}^{m}\mathfrak{M} - {}^{n}\overset{+1}{\mathfrak{M}}\,{}^{m+1}\mathfrak{M} + {}^{n}\overset{+2}{\mathfrak{M}}\,{}^{m+2}\mathfrak{M} - \ldots \pm {}^{n}\overset{+k}{\mathfrak{M}}\,{}^{m+k}\mathfrak{M} \mp \ldots \pm {}^{n}\mathfrak{N}\,{}^{n}\mathfrak{M} = 0. $$

ubi n et m sunt numeri integri positivi et $n > m$.

DEMONSTRATIO.

Constat esse

$$ (1+x)^{-m-1} = 1 - {}^{m+1}\mathfrak{A}x + {}^{m+2}\mathfrak{B}x^2 - {}^{m+3}\mathfrak{C}x^3 + \ldots \pm \overset{-m}{\mathfrak{N}}x^{n-m} \ldots $$

$$ \text{et } (1+x)^{n} = 1 + {}^{n}\mathfrak{A}x + {}^{n}\mathfrak{B}x^2 + {}^{n}\mathfrak{N}x^3 + \ldots + \overset{-m}{\mathfrak{N}}x^{n-m} \ldots $$

Ex his seriebus per multiplicationem sequitur:

$$
\begin{array}{l|l|l|l|l}
(1+x)^{n-m-1} = 1 - 1\,{}^{m+1}\mathfrak{A} & x + 1\,{}^{m+2}\mathfrak{B} & x^2 - 1\,{}^{m+3}\mathfrak{C} & x^3 + \ldots\ldots \pm 1\,{}^{n}\overset{-m}{\mathfrak{N}} & x^{n-m}\ldots \\
\qquad + {}^{n}\mathfrak{A}\,1 & - {}^{n}\mathfrak{A}\,{}^{m+1}\mathfrak{A} & + {}^{n}\mathfrak{A}\,{}^{m+2}\mathfrak{B} & \mp {}^{n}\mathfrak{A}\,{}^{n-1}\overset{-m-1}{\mathfrak{N}} & \\
 & + {}^{n}\mathfrak{B}\,1 & - {}^{n}\mathfrak{B}\,{}^{m+1}\mathfrak{A} & \pm {}^{n}\mathfrak{B}\,{}^{n-2}\overset{-m-2}{\mathfrak{N}} & \\
 & & + {}^{n}\mathfrak{C}\,1 & \mp {}^{n}\mathfrak{C}\,{}^{n-3}\overset{-m-3}{\mathfrak{N}} & \\
 & & & \quad '' \quad '' & \\
 & & & \mp {}^{n}\mathfrak{K}\,{}^{n-k}\overset{-m-k}{\mathfrak{N}} & \\
 & & & \quad '' \quad '' & \\
 & & & - {}^{n}\overset{-m-1}{\mathfrak{N}}\,{}^{m+1}\mathfrak{A} & \\
 & & & + {}^{n}\overset{-m}{\mathfrak{N}}\,1 &
\end{array}
$$

Alteram formam eiusdem potentiae ipsa formula binominalis affert:

$$ (1+x)^{n-m-1} = 1 + {}^{n-m-1}\mathfrak{A}x + {}^{n-m-1}\mathfrak{B}x^2 + {}^{n-m-1}\mathfrak{C}x^3 + \ldots {}^{n-m-1}\mathfrak{N}x^{n-m} \ldots $$

et quoniam in utraque forma coefficientes iisdem literae x potentiis annexos aequales esse oportet, ex posteriori binomii $(1+x)^{n-m-1}$ evolutione, utriusque terminum $(n-m+1)^{tum}$ cum sequentibus evanescere, patet et reperitur esse:

$\pm 1\,{}^{n}\mathfrak{M}$

$$\left.\begin{array}{l} \pm\, 1 \;\; {}^{n}\mathfrak{N}^{-m} \\ \mp\, {}^{n}\mathfrak{A}\, {}^{n-1}\mathfrak{N}^{-m-1} \\ \pm\, {}^{n}\mathfrak{B}\, {}^{n-2}\mathfrak{N}^{-m-2} \\ \quad '' \\ \quad '' \\ \pm\, {}^{n}\mathfrak{K}\, {}^{n-k}\mathfrak{N}^{-m-k} \\ \quad '' \\ \quad '' \\ -\, {}^{n}\mathfrak{M}^{-m-1}\, {}^{n+1}\mathfrak{A} \\ +\, {}^{n}\mathfrak{N}^{-m} \;\; 1 \end{array}\right\} = {}^{n-m-1}\mathfrak{N}^{-m} = o \text{ et etiam } \left.\begin{array}{l} \pm\, {}^{n}\mathfrak{N} \;\; {}^{n}\mathfrak{M} \\ \mp\, {}^{n}\mathfrak{N}^{-1}\, {}^{n-1}\mathfrak{M} \\ \pm\, {}^{n}\mathfrak{N}^{-2}\, {}^{n-2}\mathfrak{M} \\ \quad '' \\ \quad '' \\ \pm\, {}^{n}\mathfrak{N}^{-k}\, {}^{n-k}\mathfrak{M} \\ \quad '' \\ \quad '' \\ -\, {}^{n}\mathfrak{M}^{+1}\, {}^{m+1}\mathfrak{M} \\ +\, {}^{n}\mathfrak{M} \;\; {}^{m}\mathfrak{M} \end{array}\right\} = o,$$

propter relationem enim novissimam

$${}^{q}\mathfrak{P} = {}^{q}\mathfrak{Q}^{-p}$$

est

$${}^{n}\mathfrak{K} = {}^{n}\mathfrak{N}^{-k}$$

$${}^{n-k}\mathfrak{M} = {}^{n-k}\mathfrak{N}^{-k-m}$$

$${}^{n}\mathfrak{M} = {}^{n}\mathfrak{N}^{-m}$$

§. 3.

THEOREMA.

Termini primi numero $(n+1)$ feriei cuiuslibet in ferie verticali inftruantur, iisque per multiplicationem iungantur termini refpondentes feriei $1 - {}^{n}\mathfrak{A} + {}^{n}\mathfrak{B} + \ldots\ldots \pm {}^{n}\mathfrak{N} = (1-1)^{n}$ et prodeuntium productorum fummam exprimat $\pm\,\nu$, tum ν erit terminus primus differentiarum $(n)^{tarum}$ feriei datae.

DEMONSTRATIO.

Ex praefcripto haec producta accipimus:

$$\begin{array}{l} +\, 1\, [z] \\ -\, {}^{n}\mathfrak{A}[z + {}^{1}\mathfrak{A}\alpha] \\ +\, {}^{n}\mathfrak{B}[z + {}^{2}\mathfrak{A}\alpha + {}^{2}\mathfrak{B}\beta] \\ \quad '' \qquad '' \qquad '' \\ \pm\, {}^{n}\mathfrak{M}[z + {}^{m}\mathfrak{A}\alpha + {}^{m}\mathfrak{B}\beta \;\ldots\ldots\; + {}^{m}\mathfrak{M}\mu] \\ \mp\, {}^{n}\mathfrak{M}^{+1}[z + {}^{m+1}\mathfrak{A}\alpha + {}^{m+1}\mathfrak{B}\beta \;\ldots\ldots\; + {}^{m+1}\mathfrak{M}\mu + {}^{m+1}\mathfrak{M}^{+1}\mu^{+1}] \\ \quad '' \qquad '' \qquad '' \qquad\qquad '' \qquad '' \\ \mp\, {}^{n}\mathfrak{N}^{-1}[z + {}^{n-1}\mathfrak{A}\alpha + {}^{n-1}\mathfrak{B}\beta \;\ldots\ldots\; + {}^{n-1}\mathfrak{M}\mu + {}^{n-1}\mathfrak{M}^{+1}\mu^{+1} \;\ldots\ldots\; {}^{n-1}\mathfrak{N}^{-1}\nu^{-1}] \\ \pm\, {}^{n}\mathfrak{N}[z + {}^{n}\mathfrak{A}\alpha + {}^{n}\mathfrak{B}\beta \;\ldots\ldots\; + {}^{n}\mathfrak{M}\mu + {}^{n}\mathfrak{M}^{+1}\mu^{+1} \;\ldots\ldots\; {}^{n}\mathfrak{N}^{-1}\nu^{-1} + {}^{n}\mathfrak{N}\nu] \end{array}$$

Multiplicatione revera peracta, provenientium columnarum $(m+1)^{ta}$ hos terminos complectitur:

C $\qquad\qquad \pm[+{}^{n}\mathfrak{M}$

$$\pm[+{}^n\mathfrak{M}\,{}^m\mathfrak{M} - {}^n\overset{+1}{\mathfrak{M}}\,{}^{m+1}\mathfrak{M} + \ldots \mp {}^n\overset{-1}{\mathfrak{N}}\,{}^{n-1}\mathfrak{M} \pm {}^n\mathfrak{N}\,{}^n\mathfrak{M}]\,\mu,$$

in quibus ſuperiora vel inferiora ſigna valent, intra uncos pro numeri (n), extra uncos pro numeri (m) paritate vel imparitate. Itaque $(m+1)^{ta}$ columna eosdem terminos continet, quorum ſummam nihilo aequari, vidimus §. 2. eamque ob cauſam in colligendis valoribus productorum appoſitorum praeter ultimam columnam ad dextram, quae termino conſtat uno, omnes evaneſcunt, et ipſa productorum ſumma eſt: $\pm{}^n\mathfrak{N}\,{}^n\mathfrak{N}\,\nu = \pm\nu$, quae ſimul primum terminum differentiarum $(n)^{tarum}$ ſeriei arithmeticae datae exprimit.

§. 4.

THEOREMA.

Termini ${}^{mk}\mathfrak{N}$; ${}^{(m+1)k}\mathfrak{N}$; ${}^{(m+2)k}\mathfrak{N}$; .... ${}^{(m+r)k}\mathfrak{N}$; .... ſerie arithmetica ordinis $(n)^{ti}$ progrediuntur, cuius differentiae noviſſimae aequant k^n.

DEMONSTRATIO.

Coefficientes binominales propoſiti multiplicentur per 1, 2, 3, 4 .... n evolvanturque producta, quibus eorum numeratores conſtant, accipiemus:

$1.2.3\ldots n\;{}^{mk}\mathfrak{N} = m^n$	$k^n - m^{n-1}$	$Ak^{n-1} + m^{n-2}$	$Bk^{n-2} - \ldots \pm m^2$	$\overset{-2}{N}k^2 \mp m$	$\overset{-1}{N}k$
$1.2.3\ldots n\;{}^{(m+1)k}\mathfrak{N} = (m+1)^n$	$-(m+1)^{n-1}$	$+(m+1)^{n-2}$	$\pm(m+1)^2$	$\mp(m+1)$	
$1.2.3\ldots n\;{}^{(m+2)k}\mathfrak{N} = (m+2)^n$	$-(m+2)^{n-1}$	$+(m+2)^{n-2}$	$\pm(m+2)^2$	$\mp(m+2)$	
$1.2.3\ldots n\;{}^{(m+3)k}\mathfrak{N} = (m+3)^n$	$-(m+3)^{n-1}$	$+(m+3)^{n-2}$	$\pm(m+3)^2$	$\mp(m+3)$	
″ ″	″	″	″	″	
″ ″	″	″	″	″	
$1.2.3\ldots n\;{}^{(m+r)k}\mathfrak{N} = (m+r)^n$	$-(m+r)^{n-1}$	$+(m+r)^{n-2}$	$\pm(m+r)^2$	$\mp(m+r)$	
″ ″	″	″	″	″	
″ ″	″	″	″	″	

Index (1. 2. 3. 4 .... [n−1]).

Columna prima ad dextram ſignorum aequalitatis eſt ſeries arithmetica ordinis $(n)^{ti}$, et quaelibet poſteriorum ordinis inferioris, itaque ſummae terminorum homologorum ex omnibus columnis ſeriem conſtituunt, quae cum prima columna eundem ordinem $(n)^{tum}$, easdemque differentias noviſſimas $= 1.2.3\ldots\ldots n.k^n$ communes habet (v. *Additamentorum ad theoriam ſerierum arithmeticarum*, §. 4. et §. 7. Coroll. Exempl. III.). Terminis omnibus, ut ad ſeriem datam perveniamus, per $1.2.3.4\ldots(n-1)n$ diviſis, manet quidem ordo ſeriei $(n)^{tus}$, ſed noviſſimae differentiae fiunt $= k^n$.

COROL-

COROLLARIUM.

Posito $m = 0$, coefficientes binominales seriei datae abeunt in hos:

$$0;\ {}^{k}\mathfrak{N};\ {}^{2k}\mathfrak{N};\ {}^{3k}\mathfrak{N};\ \ldots\ {}^{rk}\mathfrak{N};\ \ldots$$

quae series ordinem priorem $(n)^{tum}$, nec non differentias novissimas k^{n} retinet, atque secundum §. 3. differentiarum $(s)^{tarum}$ terminum primum habet:

$$= 1.0 - {}^{s}\mathfrak{A}\,{}^{k}\mathfrak{N} + {}^{s}\mathfrak{B}\,{}^{2k}\mathfrak{N} - {}^{s}\mathfrak{C}\,{}^{3k}\mathfrak{N} + \ldots \pm {}^{s}\mathfrak{S}\,{}^{sk}\mathfrak{N}$$

quorum terminorum summa pro $s > n$ evanescit, quia differentias $(n)^{tas}$ insecuturae nullae sunt, sed pro $s = n$ novissimarum differentiarum valorem exprimit, quem esse k^{n} in theoremate didicimus.

SCHOLION. Signis A'; B'; C'; .... $\overset{-1}{N'}$; indicantur classes singulae combinationum simpliciter ex elementis 1, 2, 3 .... $(n-1)$ conflatarum, omissis repetitionibus.

§. 5.

THEOREMA.

Termini aequidistantes cuiuslibet seriei arithmeticae ordinis $(p)^{ti}$ seriem eiusdem ordinis constituunt.

DEMONSTRATIO.

Seriem arithmeticam P ordinis $(p)^{ti}$ §. 2. *Additamentorum* nostrorum vidimus sic exhiberi posse:

$$\begin{aligned}
P = &\ z \\
&+[z + {}^{1}\mathfrak{A}\alpha] \\
&+[z + {}^{2}\mathfrak{A}\alpha + {}^{2}\mathfrak{B}\beta] \\
&\quad '' \quad '' \quad '' \\
&+[z + {}^{p}\mathfrak{A}\alpha + {}^{p}\mathfrak{B}\beta + \ldots + {}^{p}\mathfrak{P}\pi] \\
&+[z + {}^{p+1}\mathfrak{A}\alpha + {}^{p+1}\mathfrak{B}\beta + \ldots + {}^{p+1}\mathfrak{P}\pi] \\
&\quad '' \quad '' \quad '' \quad ''
\end{aligned}$$

Terminus $(n+1)$ huius seriei indicetur per signum $7(n+1)$, erit:

$$\left.\begin{aligned}
7\ \ 1 &= z + 0\alpha + 0\beta + \ldots + 0\pi \\
7(k+1) &= z + {}^{1k}\mathfrak{A}\alpha + {}^{1k}\mathfrak{B}\beta + \ldots + {}^{1k}\mathfrak{P}\pi \\
7(2k+1) &= z + {}^{2k}\mathfrak{A}\alpha + {}^{2k}\mathfrak{B}\beta + \ldots + {}^{2k}\mathfrak{P}\pi \\
7(3k+1) &= z + {}^{3k}\mathfrak{A}\alpha + {}^{3k}\mathfrak{B}\beta + \ldots + {}^{3k}\mathfrak{P}\pi \\
'' \quad & \quad '' \quad '' \quad '' \quad '' \\
7(mk+1) &= z + {}^{mk}\mathfrak{A}\alpha + {}^{mk}\mathfrak{B}\beta + \ldots + {}^{mk}\mathfrak{P}\pi \\
'' \quad & \quad '' \quad '' \quad '' \quad ''
\end{aligned}\right\} = Q$$

In his terminis, quos per Q designavimus, ultima columna ad dextram est series ordinis $(p)^{ti}$, reliquae columnae sunt inferiorum ordinum, uti ex

§. 4.

§. 4. Corollario intelligitur, itaque series ipsa Q ad eundem ordinem adscendit, id quod in *Additamentis* §. 4. *Corollar*. ostendimus. Sed iidem termini Q in serie P aequidistant, quoniam binis quibusque terminorum Q in serie P interiecti sunt termini numero $(k-1)$.

§. 6.

P R O B L E M A.

Invenire terminos primos differentiarum a serie Q §. 5. proposita oriundarum, quae constat ex terminis aequidistantibus seriei arithmeticae P ordinis $(p)^{ti}$.

S O L U T I O.

In differentiis a serie Q derivandis designetur terminus primus differentiarum $(m)^{tarum}$ per Δ^m, invenientur:

secundum §. 3. aequationes sequentes:

$$-\Delta^1 = 1\,[z + 0\alpha + 0\beta \,..\, + 0\pi] \\ -{}^1\mathfrak{A}[z + {}^{1k}\mathfrak{A}\alpha + {}^{1k}\mathfrak{B}\beta \,..\, + {}^{1k}\mathfrak{B}\pi]$$

$$+\Delta^2 = 1\,[z + 0\alpha + 0\beta \,..\, + 0\pi] \\ -{}^2\mathfrak{A}[z + {}^{1k}\mathfrak{A}\alpha + {}^{1k}\mathfrak{B}\beta \,..\, + {}^{1k}\mathfrak{P}\pi] \\ +{}^2\mathfrak{B}[z + {}^{2k}\mathfrak{A}\alpha + {}^{2k}\mathfrak{B}\beta \,..\, + {}^{2k}\mathfrak{P}\pi]$$

$$\pm\Delta^m = 1\,[z + 0\alpha \,...\, 0\mu \,..\, + 0\pi] \\ -{}^m\mathfrak{A}[z + {}^{1k}\mathfrak{A}\alpha \,..\, {}^{1k}\mathfrak{M}\mu \,..\, + {}^{1k}\mathfrak{P}\pi] \\ +{}^m\mathfrak{B}[z + {}^{2k}\mathfrak{A}\alpha \,..\, {}^{2k}\mathfrak{M}\mu \,..\, + {}^{2k}\mathfrak{P}\pi] \\ 〃 \quad 〃 \quad 〃 \quad 〃 \quad 〃 \\ \pm{}^m\mathfrak{M}[z + {}^{mk}\mathfrak{A}\alpha \,..\, {}^{mk}\mathfrak{M}\mu \,..\, + {}^{mk}\mathfrak{P}\pi]$$

$$\pm\Delta^p = 1\,[z + 0\alpha \,...\, + 0\overset{-1}{\pi} + 0\pi] \\ -{}^m\mathfrak{A}[z + {}^{1k}\mathfrak{A}\alpha \,...\, + {}^{1k}\overset{-1}{\mathfrak{P}}\overset{-1}{\pi} + {}^{1k}\mathfrak{P}\pi] \\ +{}^m\mathfrak{B}[z + {}^{2k}\mathfrak{A}\alpha \,...\, + {}^{2k}\overset{-1}{\mathfrak{P}}\overset{-1}{\pi} + {}^{2k}\mathfrak{P}\pi] \\ 〃 \quad 〃 \quad 〃 \quad 〃 \quad 〃 \\ \pm{}^p\mathfrak{P}[z + {}^{pk}\mathfrak{A}\alpha \,...\, + {}^{pk}\overset{-1}{\mathfrak{P}}\overset{-1}{\pi} + {}^{pk}\mathfrak{P}\pi]$$

et reductione secundum §. 4. Corollarium peracta:

$$\Delta^1 = k\alpha + {}^1\mathfrak{A}[{}^k\mathfrak{B}\beta + {}^k\mathfrak{C}\gamma \,..\, + {}^k\mathfrak{P}\pi]$$

$$\Delta^2 = k^2\beta - {}^2\mathfrak{A}[{}^k\mathfrak{C}\gamma + {}^k\mathfrak{D}\delta \,..\, + {}^k\mathfrak{P}\pi] \\ + {}^2\mathfrak{B}[{}^{2k}\mathfrak{C}\gamma + {}^{2k}\mathfrak{D}\delta \,..\, + {}^{2k}\mathfrak{P}\pi]$$

$$\pm\Delta^m = \pm k^m\mu - {}^m\mathfrak{A}[{}^k\overset{+1}{\mathfrak{M}}\overset{+1}{\mu} \,..\, + {}^k\mathfrak{P}\pi] \\ + {}^m\mathfrak{B}[{}^{2k}\overset{+1}{\mathfrak{M}}\overset{+1}{\mu} \,..\, + {}^{2k}\mathfrak{P}\pi] \\ 〃 \quad 〃 \quad 〃 \\ \pm{}^m\mathfrak{M}[{}^{mk}\overset{+1}{\mathfrak{M}}\overset{+1}{\mu} \,..\, + {}^{mk}\mathfrak{P}\pi]$$

$$\Delta^p = k^p\pi$$

§. 7.

§. 7.

P R O B L E M A.

De ſerie arithmetica ordinis p^{ti}, qualem exhibet *P* §. 5. praeter terminum primum datus eſt terminus $(k+1)^{tus}$, $(2k+1)^{tus}$, $(3k+1)^{tus}$ . . . $(pk+1)^{tus}$, quaeritur integrae reſtitutio, interiectis quidem $(k-1)$ terminis inter binos quosque terminos datorum.

S O L V T I O.

Problematis ſolutio generalis valoribus ipſis, quos primi differentiarum termini Δ^p; Δ^{p-1}; Δ^{p-2}; . . . Δ^3; Δ^2; Δ^1; ſecundum §. 6. habent, affertur, ex illis enim aequationibus quantitatum π; $\overset{-1}{\pi}$; $\overset{-2}{\pi}$; . . . . γ; β; α; valores erui poſſunt, iisque repertis, expedita adeſt ipſius ſeriei *P* reſtitutio, quum primus eius terminus ſit e numero datorum. Sed quoniam, ſi ad ipſam interpolationem ſeu reſtitutionem ſeriei cuiusdam redeat tandem labor, nullius ſunt utilitatis formulae generaliores, ſperavimus illis relictis non ingratum fore lectoribus quarundam ſerierum ordinis certi reſtitutionem tabula parata adiuvare, quae §. ſequenti exhibetur.

§. 8.

PROBLEMATIS §. 7. PROPOSITI VSVS IN INTERPOLANDIS SERIEBVS ARITHMETICIS ORDINVM SEPTIMO INFERIORVM OSTENDITVR.

Interpolantur fere arithmeticae ſeries, quae ordinem ſextum non transcendunt, quia cum ordine ſeriei interpolandae interpolationis ipſius augetur labor, et poſteriores differentiae, ſi quae adſunt, ſine incommodo plerumque negligi poſſunt. Subſtitimus igitur in ipſis ſeriebus ordinis ſexti et tabulam, cuius ope in ſerie arithmetica ordinis ſeptimo inferioris inter binos quosque terminos termini numero $k-1=1$, 2, 3, 4, 5, 6, facilius interiici poſſint, computavimus, ne in qualibet interpolatione iidem calculi denuo ſint repetendi. Tabula ipſa accommodata eſt ſeriei *P* §. 5. exhibitae et aequationibus §. 6. omiſſis quidem differentiis, quae ſextas inſequuntur. Termini ſcilicet primi differentiarum 1^{rum}, 2^{rum}, 3^{rum}, 4^{rum}, 5^{rum}, 6^{rum}, ſignis Δ^1, Δ^2, Δ^3, Δ^4, Δ^5, Δ^6, eorumque valores ſequentibus exprimuntur aequationibus:

pro

pro k=2

$$\begin{aligned}
\Delta^{1} &= 2\alpha + 1\beta \\
\Delta^{2} &= \cdot\ 4\beta + 4\gamma + 1\delta \\
\Delta^{3} &= \cdot\ \cdot\ 8\gamma + 12\delta + 6\varepsilon + 1\zeta \\
\Delta^{4} &= \cdot\ \cdot\ \cdot\ 16\delta + 32\varepsilon + 24\zeta \\
\Delta^{5} &= \cdot\ \cdot\ \cdot\ \cdot\ 32\varepsilon + 80\zeta \\
\Delta^{6} &= \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ 64\zeta
\end{aligned}$$

pro k=3

$$\begin{aligned}
\Delta^{1} &= \text{[illegible]} + 3\beta + 1\gamma \\
\Delta^{2} &= \cdot\ 9\beta + 18\gamma + 15\delta + 6\varepsilon + 1\zeta \\
\Delta^{3} &= \cdot\ \cdot\ 27\gamma + 81\delta + 108\varepsilon + 81\zeta \\
\Delta^{4} &= \cdot\ \cdot\ \cdot\ 81\delta + 324\varepsilon + 594\zeta \\
\Delta^{5} &= \cdot\ \cdot\ \cdot\ \cdot\ 243\varepsilon + 1215\zeta \\
\Delta^{6} &= \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ 729\zeta
\end{aligned}$$

pro k=4

$$\begin{aligned}
\Delta^{1} &= 4\alpha + 6\beta + 4\gamma + 1\delta \\
\Delta^{2} &= \cdot\ 16\beta + 48\gamma + 68\delta + 56\varepsilon + 28\zeta \\
\Delta^{3} &= \cdot\ \cdot\ 64\gamma + 288\delta + 624\varepsilon + 480\zeta \\
\Delta^{4} &= \cdot\ \cdot\ \cdot\ 256\delta + 1536\varepsilon + 4480\zeta \\
\Delta^{5} &= \cdot\ \cdot\ \cdot\ \cdot\ 1024\varepsilon + 7680\zeta \\
\Delta^{6} &= \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ 6096\zeta
\end{aligned}$$

pro k=5

$$\begin{aligned}
\Delta^{1} &= 5\alpha + 10\beta + 10\gamma + 5\delta + 1\varepsilon \\
\Delta^{2} &= \cdot\ 25\beta + 100\gamma + 200\delta + 250\varepsilon + 210\zeta \\
\Delta^{3} &= \cdot\ \cdot\ 125\gamma + 750\delta + 2250\varepsilon + 4375\zeta \\
\Delta^{4} &= \cdot\ \cdot\ \cdot\ 625\delta + 5000\varepsilon + 20000\zeta \\
\Delta^{5} &= \cdot\ \cdot\ \cdot\ \cdot\ 3125\varepsilon + 31250\zeta \\
\Delta^{6} &= \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ 15625\zeta
\end{aligned}$$

pro k=6

$$\begin{aligned}
\Delta^{1} &= 6\alpha + 15\beta + 20\gamma + 15\delta + 6\varepsilon + 1\zeta \\
\Delta^{2} &= \cdot\ 36\beta + 180\gamma + 465\delta + 780\varepsilon + 922\zeta \\
\Delta^{3} &= \cdot\ \cdot\ 216\gamma + 1620\delta + 6210\varepsilon + 15795\zeta \\
\Delta^{4} &= \cdot\ \cdot\ \cdot\ 1296\delta + 12960\varepsilon + 65880\zeta \\
\Delta^{5} &= \cdot\ \cdot\ \cdot\ \cdot\ 7776\varepsilon + 97200\zeta \\
\Delta^{6} &= \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ 46656\zeta
\end{aligned}$$

pro k=7

$$\begin{aligned}
\Delta^{1} &= 7\alpha + 21\beta + 35\gamma + 35\delta + 21\varepsilon + 7\zeta \\
\Delta^{2} &= \cdot\ 49\beta + 294\gamma + 931\delta + 1960\varepsilon + 2989\zeta \\
\Delta^{3} &= \cdot\ \cdot\ 343\gamma + 3087\delta + 14406\varepsilon + 45276\zeta \\
\Delta^{4} &= \cdot\ \cdot\ \cdot\ 2401\delta + 28812\varepsilon + 177674\zeta \\
\Delta^{5} &= \cdot\ \cdot\ \cdot\ \cdot\ 16807\varepsilon + 252105\zeta \\
\Delta^{6} &= \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ 117649\zeta
\end{aligned}$$

EXEMPLVM I. Interiiciendi sint termini terni binis quibusque seriei $1^2, 2^2, 3^2, 4^2, 5^2 \ldots\ldots$ In hoc exemplo numerus terminorum interiiciendorum est $k-1=3$, seriei interpolandae ordo secundus, et $\Delta^{1}=3$, $\Delta^{2}=2$, ergo $k=4$ et $\gamma, \delta, \varepsilon, \zeta=0$, unde sequitur:

$$\Delta^{1} = 4\alpha + 6\beta = 3$$
$$\Delta^{2} = \qquad 6\beta = 2$$

quibus ex aequationibus reperitur:

$$\beta = \tfrac{1}{8} \text{ et } \alpha = \tfrac{9}{16}$$

Series itaque proposita post interpolationem hanc accipit formam:

1	h. e. reductione facta:	$\frac{16}{16} = 1$
$+(1 + 1\tfrac{9}{16})$		$\frac{25}{16}$
$+(1 + 2\tfrac{9}{16} + 1\tfrac{1}{8})$		$\frac{36}{16}$
$+(1 + 3\tfrac{9}{16} + 3\tfrac{1}{8})$		$\frac{49}{16}$
$+(1 + 4\tfrac{9}{16} + 6\tfrac{1}{8})$		$\frac{64}{16} = 4$
$+(1 + 5\tfrac{9}{16} + 10\tfrac{1}{8})$		$\frac{81}{16}$
$+(1 + 6\tfrac{9}{16} + 15\tfrac{1}{8})$		$\frac{100}{16}$
$+(1 + 7\tfrac{9}{16} + 21\tfrac{1}{8})$		$\frac{121}{16}$
$+(1 + 8\tfrac{9}{16} + 28\tfrac{1}{8})$		$\frac{144}{16} = 9$
$+(1 + 9\tfrac{9}{16} + 35\tfrac{1}{8})$		$\frac{167}{16}$
" " "		"

EXEM-

EXEMPLVM II. Interdum primus feriei interpolandae terminus eſt (o), v. c. in ferie $0,\ 1^2,\ 2^2,\ 3^2,\ 4^2\ \ldots\ldots$ Tum hoc ordine calculi inſtituuntur, tribus iterum terminis interiectis:

Termini feriei interpolandae ſunt:

$$0 = 0$$
$$1 = 0 + 1 \cdot 1$$
$$4 = 0 + 2 \cdot 1 + 1 \cdot 2$$
$$9 = 0 + 3 \cdot 1 + 3 \cdot 2$$

itaque

$$\Delta^1 = 1 \qquad \Delta^2 = 16\beta = 2$$
$$\Delta^2 = 2 \qquad \beta = \tfrac{1}{8}$$
$$\gamma, \delta \ldots = 0 \qquad \Delta^1 = 4\alpha + 6\beta = 1$$
$$k = 4 \qquad 4\alpha + \tfrac{3}{4} = 1$$
$$\alpha = \tfrac{1}{16}$$

et ipſa feries poſt interpolationem peractam:

$$0 \qquad \text{h. e.} \quad 0 = 0$$
$$+0 + 1\tfrac{1}{16} \qquad \tfrac{1}{16}$$
$$+0 + 2\tfrac{1}{16} + 1\tfrac{1}{8} \qquad \tfrac{4}{16}$$
$$+0 + 3\tfrac{1}{16} + 3\tfrac{1}{8} \qquad \tfrac{9}{16}$$
$$+0 + 4\tfrac{1}{16} + 6\tfrac{1}{8} \qquad \tfrac{16}{16} = 1$$
$$+0 + 5\tfrac{1}{16} + 10\tfrac{1}{8} \qquad \tfrac{25}{16}$$
$$\text{〃} \qquad \text{〃} \qquad \text{〃} \qquad \text{〃}$$

EXEMPLVM III. Series ordinis tertii $3,\ 4,\ 9,\ 20,\ 39,\ 68,\ 109\ldots$ interiectis terminis quaternis, ſit interpolanda. In hoc exemplo eſt:

$$k = 5,\ \Delta^1 = 2,\ \Delta^2 = 4,\ \Delta^3 = 1,\ \text{et } \delta, \varepsilon, \zeta \ldots = 0;$$

itaque:

$$125\gamma = \Delta^3 = 2 \qquad \text{et:} \quad \gamma = \tfrac{2}{125}$$
$$25\beta + 100\gamma = \Delta^2 = 4 \qquad \beta = \tfrac{12}{125}$$
$$5\alpha + 10\beta + 10\gamma = \Delta^1 = 0 \qquad \gamma = \tfrac{3}{125}$$

et feries interpolata:

$$3$$
$$3 - 1\tfrac{3}{125} +$$
$$3 - 2\tfrac{3}{125} + 1\tfrac{12}{125}$$
$$3 - 3\tfrac{3}{125} + 3\tfrac{12}{125} + 1\tfrac{2}{125}$$
$$3 - 4\tfrac{3}{125} + 6\tfrac{12}{125} + 4\tfrac{2}{125}$$
$$3 - 5\tfrac{3}{125} + 10\tfrac{12}{125} + 10\tfrac{2}{125}$$
$$\text{〃} \qquad \text{〃} \qquad \text{〃} \qquad \text{〃}$$

ubi terminorum denominator communis eſt 125, numeratores vero ſunt:

375
372
381
404
443
500
〃

§. 9.

Ad commoda interpolationis noſtrae hoc inprimis referendum eſt, quod, ſimulac reperti ſunt valores literarum $\alpha, \beta, \gamma \ldots$, ordine quidem inverſo, reliqua omnia numerorum figuratorum ope peraguntur, id quod aliter ſeſe in interpolatione vulgari habet. Nec tamen alterutram methodorum aut ubique adhibendam, aut omnino excludendam, ſed ex ipſo negotio peragendo diiudicandum eſſe, contendimus, qua potius utamur. Nempe in vulgari interpolatione ſerierum arithmeticarum implicamur evolutione coefficientium binomialium a potentiis fractis oriundarum, noſtra vero eliminationem incognitarum $\alpha, \beta, \gamma, \delta \ldots$ adducit, quorum laborum ille cum numero terminorum computandorum, hic cum ordine ſeriei interpolandae creſcit. Itaque conſultius

fultius forsitan videbitur, si pauci terminorum seriei interpolatae quaerantur, veterem, sin autem maior eorum numerus, vel omnes exhibendi sint, nostram methodum sequi. Ne vero quidquam desideretur et in promptu sit utriusque methodi comparatio, duplici interpolatione binis quibusque terminis seriei tertii ordinis 5, 50, 320, 1065, 2536 ... terminos quaternos interiiciemus. Hoc in exemplo est $k=5$, $\Delta^1=45$, $\Delta^2=225$, $\Delta^3=250$:

Series igitur interpolatione vulgari peracta hanc faciem accipit:

Nostra methodo ex aequationibus §. 8. propositis:

$$125\gamma=\Delta^3=250 \quad \text{sequitur:} \quad \gamma=2$$
$$25\beta+100\gamma=\Delta^2=225 \quad \beta=1$$
$$5\alpha+10\beta+10\gamma=\Delta^1=45 \quad \alpha=1,$$

et alteram series interpolata formam induit:

5	5	5
5 + $\frac{1}{5}$𝔄45 + $\frac{1}{5}$𝔅225 + $\frac{1}{5}$ℭ250	5 + 1.3	8
5 + $\frac{2}{5}$𝔄45 + $\frac{2}{5}$𝔅225 + $\frac{2}{5}$ℭ250	5 + 2.3 + 1.2	12
5 + $\frac{3}{5}$𝔄45 + $\frac{3}{5}$𝔅225 + $\frac{3}{5}$ℭ250	5 + 3.3 + 3.2 + 1.2	19
5 + $\frac{4}{5}$𝔄45 + $\frac{4}{5}$𝔅225 + $\frac{4}{5}$ℭ250	5 + 4.3 + 6.2 + 4.2	31
5 + $\frac{5}{5}$𝔄45 + $\frac{5}{5}$𝔅225 + $\frac{5}{5}$ℭ250	5 + 5.3 + 10.2 + 10.2	50
5 + $\frac{6}{5}$𝔄45 + $\frac{6}{5}$𝔅225 + $\frac{6}{5}$ℭ250	5 + 6.3 + 15.2 + 10.2	78
" " " "	" " " "	"

Sed utraque series reductione facta numeris iisdem constat:

Vbi plures termini seriei interpolatae desiderantur, nostram interpolationem vulgari paullo ante praetulimus, nihilo tamen minus fieri potest, ut sive auxilium tabulae nostrae §. 8. propositae detrectetur, sive eius limites ipsos series data vel ordine suo, vel numero terminorum interiiciendorum transgrediatur. Quo facto, de prioribus terminis totidem computentur more consueto, quot ad literarum α, β, γ, δ, ε ... valores inveniendos sufficiant, et posteriores termini nostra interpolatione non sine aliquo calculorum compendio poterunt proferri, sic enim et fractionum molestiae per numeros figuratos, et tabularum paratarum angustiae per literarum α, β, γ, δ ..... valores directe repertos vitantur.

III.

FVNCTIONES

LOGARITHMICAE ET TRIGONOMETRICAE

IN SERIES INFINITAS SOLVTAE.

FUNCTIONES LOGARITHMICAE ET TRIGONOMETRICAE IN SERIES INFINITAS SOLUTAE.

§. 1.

Functiones logarithmicas et trigonometricas in series infinitas ita solvit L. EULERUS *) (v. *Introductio in Analysin infinitorum*, Lib. I. Cap. VII. et VIII.) ut miremur quidem et artem et acumen Analystae, nihilo tamen minus tacito optemus pectore et illis artificiis carere et ad easdem series facilius deduci posse. Aliquo certe molestiae et incommodi sensu affectus sum, quotiescunque illam functionum logarithmicarum et trigonometricarum in series solutionem denuo perlustrarem, mihique fuit persuasum aliam ad eas viam ducere minus tortuosam, sed ignotam adhuc et ab illa trita remotiorem. Neque spes me fefellit. Qualem enim mihi ipse aperire voluissem, talem LACROIX **) ingressum esse deprehendi in introductione operis praestantissimi: *Traité du Calcul differentiel et integral*, quod prodiit duobus tomis Lutetiae Parisiorum annis 1797 et 1798, quibus tertium volumen: *Traité des differences et des series*, anno 1800 adiecit. Is, dum alii ad illas series per functiones unius variabilis tendunt, duabus variabilibus introductis, coefficientium nodos sive minus constringit, sive felicius solvit, et rem alias arduam et difficilem deducit in planum nulloque paene labore expedit. Methodum hanc, quam ab nullo scriptorum Germanorum in illis seriebus eruendis adhibitam esse recordaremur, delectati eius et facilitate et utilitate,

 in-

*) LAGRANGIUS, (*Theorie des fonctions analytiques*, *Paris an V.* (1792.) §. 22. et 25.) HALLEIUM Britannum coniicit primum hac methodo usum esse, eamque admittit quidem, sed ei non favet.

**) Conferantur ea, quae ll. cc. protulit LAGRANGIUS.

insignibus aliquot evolutionum exemplis illustratam hoc in libello proponere non dubitavimus, sed ita, ne, Gallicum quamvis analystam sequeremur, operae tamen nostrae vestigia deessent.

§. 1.

THEOREMA.

Functione bifariam evoluta, ita quidem, ut evolutionum termini ad eiusdem variabilis potentias digerantur, eadem variabilis potentia in utraque evolutione annexa erit coefficientibus aequalibus. Sint exempli gratia P et Q functiones variabilis z et

$$P = az^{n} + bz^{n+d} + cz^{n+2d} \ldots + mz^{n+(m-1)d} + \ldots \text{ et } Q = \alpha z^{n} + \beta z^{n+d} + \gamma z^{n+2d} \ldots + \mu z^{n+(m-1)d} + \ldots$$

et $P = Q$, tum erit $m = \mu$.

DEMONSTRATIO.

Deducatur posterior series Q de priori P, erit:

$$P - Q = (a-\alpha)z^{n} + (b-\beta)z^{n+d} + (c-\gamma)z^{n+2d} + \ldots + (m-\mu)z^{n+(m-1)d} \ldots = 0$$

et sive singulum quemque terminum seriei $= P - Q$ evanescere, sive plures mutuo sese destruere oportebit. Fingamus, non singulum terminum, sed plurium evanescere summam, tum orietur aequatio inter variabilem eiusque coefficientes. Fac enim esse $(m-\mu)z^{n+(m-1)d} + (p-\pi)z^{n+(p-1)d} = 0$, adest aequatio $(m-\mu) + (p-\pi)z^{(p-m)d} = 0$, quae valorum litterae z diversitatem ad numerum $(p-m)d$ restringit, sed innumerabiles valores ei tribui non patitur, qui tamen, quod littera variabilem quantitatem notat, admittuntur. Haec argumenta ad quotlibet numerum terminorum seriei $P - Q$ transferri possunt, eoque semper redeunt, ut quantitas z non possit esse variabilis, nisi singulus quisque terminus evanescat.

COROLLARIUM.

Sint R et S functiones variabilium duarum u et x atque

$$\begin{array}{l} R = a + {}^{1}bx^{1} + {}^{1}cx^{2} \ldots + {}^{1}mx^{m-1} \ldots + {}^{1}tx^{t-1} \ldots \\ \qquad {}^{2}bu^{1} + {}^{2}cux \ldots + {}^{2}mu^{1}x^{m-2} \ldots + {}^{2}tu^{1}x^{t-2} \ldots \\ \qquad\qquad {}^{3}cu^{2} \ldots + {}^{3}mu^{2}x^{m-3} \ldots + {}^{3}tu^{2}x^{t-3} \ldots \\ \qquad\qquad\qquad '' \qquad\qquad '' \\ \qquad\qquad\qquad '' \qquad\qquad '' \\ \qquad\qquad + {}^{m}mu^{m-1} \ldots + {}^{m}tu^{m-1}x^{t-m} \ldots \\ \qquad\qquad\qquad\qquad\qquad '' \\ \qquad\qquad\qquad\qquad\qquad '' \\ \qquad\qquad\qquad\qquad\qquad {}^{t}tu^{t-1} \ldots \end{array}$$

$$\begin{array}{l} \text{et } S = \alpha + {}^{1}\beta x^{1} + {}^{1}\gamma x^{2} \ldots + {}^{1}\mu x^{m-1} \ldots + {}^{1}\tau x^{t-1} + \ldots \\ \qquad {}^{2}\beta u + {}^{2}\gamma u^{1}x \ldots + {}^{2}\mu u^{1}x^{m-1} \ldots + {}^{2}\tau u^{1}x^{t-2} \ldots \\ \qquad\qquad + {}^{3}\gamma u^{2} \ldots + {}^{3}\mu u^{2}x^{m-2} \ldots + {}^{3}\tau u^{2}x^{t-3} \ldots \\ \qquad\qquad\qquad '' \qquad\qquad '' \\ \qquad\qquad\qquad '' \qquad\qquad '' \\ \qquad\qquad {}^{m}\mu u^{m-1} \ldots + {}^{m}\tau u^{m-1}x^{t-m} \ldots \\ \qquad\qquad\qquad\qquad\qquad '' \\ \qquad\qquad\qquad\qquad\qquad '' \\ \qquad\qquad\qquad\qquad\qquad {}^{t}\tau u^{t-1} \ldots \end{array}$$

et

et $R = S$, tum ${}^{m}t = {}^{m}\tau$ erit. In ſeriebus ſcilicet R et S termini eiusdem lineae horizontalis eandem variabilis u potentiam habent, (exceptis quidem a et α) eamque ob cauſam ſeries ſecundum potentias variabilis u diſpoſitae ſunt atque hinc ſequitur (m)^{tarum} linearum horizontalium aequalitas:

$${}^{m}mu^{m-1} + {}^{m}\overset{+1}{m}.u^{m-1}x \ldots + {}^{m}tu^{m-1}x^{t-m} + \ldots = {}^{m}\mu u^{m-1} + {}^{m}\overset{+1}{\mu}u^{m-\tau}x \ldots + {}^{m}\tau u^{m-1}x^{m-\tau} + \ldots$$

et, ſublata potentia u^{m-1},

$${}^{m}m + {}^{m}\overset{+1}{m}x \ldots + {}^{m}tx^{t-m} \ldots = {}^{m}\mu + {}^{m}\overset{+1}{\mu}x \ldots + {}^{m}\tau x^{m-\tau} \ldots$$

ubi ſecundum theorema eſt ${}^{m}t = {}^{m}\tau$.

§. 3.

THEOREMA.

Eſt $p^{x} = 1 + \frac{Bx}{1} + \frac{B^{2}x^{2}}{1.2.} + \frac{B^{3}x^{3}}{1.2.3} \ldots\ldots + \frac{B^{n}x^{n}}{1.2.3..n} + \ldots\ldots$

DEMONSTRATIO.

Ponamus eſſe I. $p^{x} = A + Bx^{1} + Cx^{2} + Dx^{3} \ldots\ldots + Nx^{n-1} \ldots\ldots$
ubi $A, B, C \ldots\ldots N$ ſunt invariabiles, pariter erit:

$$\text{II. } p^{u} = A + Bu + Cu^{2} + Du^{3} \ldots\ldots + Nu^{n-1} \ldots\ldots$$

$$\text{III. } p^{x+u} = A + B(x+u) + C(x+u)^{2} + D(x+u)^{3} \ldots + N(x+u)^{n-1} \ldots\ldots$$

Ex multiplicatione ſeriei primae $= p^{x}$ per ſeriem ſecundam $= p^{u}$ oritur productum $= p^{x}p^{u}$, quod alterum ſeriei tertiae p^{x+u} valorem exhibet:

$$\begin{aligned} \text{IV. } p^{x}p^{u} = AA + ABx + ACx^{2} + ADx^{3} \ldots &+ ANx^{n-1} \ldots \\ BAu + BBux + BCux^{2} \ldots &+ B\overset{-1}{N}ux^{n-2} \\ + CAu^{2} + CBux^{2} \ldots &+ C\overset{-2}{N}u^{2}x^{n-3} \\ + DAu^{3} \ldots &+ D\overset{-3}{N}u^{3}x^{n-4} \\ &\;\;\; '' \\ &\;\;\; '' \\ &+ M\overset{-m+1}{N}u^{m-1}x^{n-m} \\ &\;\;\; '' \\ &\;\;\; '' \\ &+ NAu^{n-1} \end{aligned}$$

Terminis vero tertiae ſeriei $= p^{x+u}$ ſecundum theorema binomiale evolutis, provenit:

V. p^{x+u}

$$\text{V. } p^{x+u} = A + B\left|\begin{array}{l} x \\ u \end{array}\right. + C\left|\begin{array}{l} x^2 \\ 2ux \\ u^2 \end{array}\right. + D\left|\begin{array}{l} x^3 \\ 3ux^2 \\ 3ux^2 \\ u^3 \end{array}\right. \ldots N\left|\begin{array}{l} x^{n-1} \ldots \\ {}^{n-1}\mathfrak{A}\, u x^{n-2} \\ {}^{n-1}\mathfrak{B}\, u^2 x^{n-3} \\ {}^{n-1}\mathfrak{C}\, u^3 x^{n-4} \\ \quad '' \\ \quad '' \\ {}^{n-1}\overset{-1}{\mathfrak{M}}\, u^{m-1} x^{n-m} \\ \quad '' \\ \quad '' \\ {}^{n-1}\overset{-1}{\mathfrak{N}}\, u^{n-1} \end{array}\right.$$

Primarum linearum horizontalium evolutionis IV. et V. comparatio tantum valorem literae $A=1$ adducit, sed ex comparatione secundarum, si scribatur valor repertus 1 pro A, sequitur (§. 2. Coroll.)

$$\begin{array}{lll} Bu = Bu & \text{itaque} & B = B \\ BBux = 2Cux & & C = \frac{B^2}{1.2} \\ BCux^2 = 3Dux^2 & & D = \frac{B^3}{1.2.3} \\ BDux^3 = 4Eux^3 & & E = \frac{B^4}{1.2.3.4} \\ \quad '' & & \quad '' \\ \quad '' & & \quad '' \\ B\overset{-1}{N}ux^{n-2} = {}^{n-1}\mathfrak{A}Nux^{n-2} & & N = \frac{B^{n-1}}{1.2..(n-1)} \end{array}$$

Lex igitur, qua coefficientes procedunt, demonstrata est, hac tamen conditione, ut ea, quam seriei

$$p^x = A + Bx + Cx^2 + Dx^3 \ldots + Nx^{n-1} \ldots$$

tribuimus, forma ad res inter se pugnantes non deducat, eamque ob caussam comparatio coefficientium producti $u^{m-1}\,x^{n-m}$ instituenda est, id quod coefficientem habet

in seriei IV^{tae} columna $(n)^{ta}$

$$M\overset{-m+1}{N} = \frac{B^{m-1}}{1.2\ldots(m-1)} \cdot \frac{B^{n-m}}{1.2.3\ldots(n-m)} = \frac{B^{n-1}}{1.2\ldots(m-1).1.2\ldots(n-m)}$$

et in seriei V^{tae} columna $(n)^{ta}$

$$N^{n-1}\overset{-1}{\mathfrak{M}} = \frac{B^{n-1}}{1.2..(n-m+1)..(n-1)} \cdot \frac{(n-1).(n-2)\ldots(n-m+1)}{1\ .\ .\ 2\ \ldots\ (m-1)} = \frac{B^{n-1}}{1.2..(n-m).1.2..(m-1)}$$

quorum coefficientium aequalitate patet, formam seriei $=p^x$ adoptatam nihil involvere, quod absonum sit.

SCHOLION. De litera B, quae omnes coefficientium $C, D, E..N..$ valores ingreditur, infra (§. 5.) videbimus.

§. 4.

§. 4.

THEOREMA.

Est $\log.(1+z) = \alpha\,[z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4 + \frac{1}{5}z^5 - \frac{1}{6}z^6 + \ldots \pm \frac{1}{n}z^n \ldots]$

DEMONSTRATIO.

Sit $\log.(1+z) = \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 \ldots \nu z^n \ldots;$

erit, posito $z = x$, et deinde $z = \frac{u}{1+x}$,

$$\log.(1+x) = \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 \ldots + \overset{+1}{\nu} x^{n+1} \ldots$$

$$+ \log.(1+\frac{u}{1+x}) \qquad + \alpha\frac{u}{1+x} + \beta\frac{u^2}{(1+x)^2} + \gamma\frac{u^3}{(1+x)^3} + \delta\frac{u^4}{(1+x)^4} \ldots + \overset{+1}{\nu}\frac{u^{n+1}}{(1+x)^{n+1}} \ldots$$

Hinc denominatoribus per theorema binomiale evolutis, sequitur:

$$\text{I. } \log.(1+x) + \log.(1+\frac{u}{1+x}) = \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 \ldots + \overset{+1}{\nu} x^{n+1} \ldots$$

$$+ \alpha u - \alpha u x + \alpha u x^2 - \alpha u x^3 \ldots \pm \alpha\, {}^n\mathfrak{N} u x^n \ldots$$

$$+ \beta u^2 - \beta 2 u^2 x + \beta 3 u^2 x^2 \ldots \mp \beta\, {}^n\overset{-1}{\mathfrak{N}} u^2 x^{n-1} \ldots$$

$$+ \gamma u^3 - \gamma 3 u^3 x \ldots \pm \gamma\, {}^n\overset{-2}{\mathfrak{N}} u^3 x^{n-2} \ldots$$

$$+ \delta u^4 \ldots \mp \delta\, {}^n\overset{-3}{\mathfrak{N}} u^4 x^{n-3} \ldots$$

$$''$$

$$''$$

$$\pm \mu\, {}^n\overset{-m+1}{\mathfrak{N}} u^m x^{n-m+1} \ldots$$

$$''$$

$$''$$

$$+ \overset{+1}{\nu}\, {}^n\overset{-n}{\mathfrak{N}} u^{n+1} \ldots$$

Sed quia est

$$\text{Log.}(1+x) + \text{Log.}(1+\frac{u}{1+x}) = \text{Log.}(1+x).(1+\frac{u}{1+x}) = \text{Log.}(1+x+u),$$ evolutio I^a simul logarithmi numeri $(1+x+u)$ valorem exhibet, alterum vero eiusdem logarithmi deprehendimus, posito $z = x+u$, et potentiis binomii $(x+u)$ evolutis, scilicet:

II. log.

$$\text{II. Log.}\,(1+x+u)=\alpha\left|\begin{array}{l} x \\ u \end{array}\right. +\beta\left|\begin{array}{l} x^2 \\ 2ux \end{array}\right. +\gamma\left|\begin{array}{l} x^3 \\ 3ux^2 \\ 3u^2x \\ u^3 \end{array}\right. +\delta\left|\begin{array}{l} x^4 \\ 4ux^3 \\ 6u^2x^2 \\ 4u^3x^1 \\ u^4 \end{array}\right. \ldots\ldots \overset{+1}{\nu}\left|\begin{array}{l} x^{n+1} \ldots\ldots \\ {}^{n+1}\mathfrak{A}ux^{n} \\ {}^{n+1}\mathfrak{B}u^2x^{n-1} \\ {}^{n+1}\mathfrak{C}u^3x^{n-2} \\ {}^{n+1}\mathfrak{D}u^4x^{n-3} \\ \quad '' \\ \quad '' \\ {}^{n+1}\mathfrak{M}u^{m}x^{n-m+1} \\ \quad '' \\ \quad '' \\ {}^{n+1}\overset{+1}{\mathfrak{N}}u^{n+1} \end{array}\right.$$

Linea horizontalis in utraque evolutione Logarithmi numeri $(1+x+u)$ eadem eſt, eamque ob rem termini ſecundae lineae comparandi ſunt, ex quibus ſecundum §. 2. Corollarium ſequitur:

$$\begin{array}{lclcl} \alpha u & = & \alpha u & \text{et} & \alpha = \alpha \\ 2\beta ux & = & -\alpha ux & & \beta = -\tfrac{1}{2}\alpha \\ 3\gamma ux^2 & = & +\alpha ux^2 & & \gamma = +\tfrac{1}{3}\alpha \\ 4\delta ux^3 & = & -\alpha ux^3 & & \delta = -\tfrac{1}{4}\alpha \\ '' & & '' & & '' \\ (n+1)\overset{+1}{\nu}ux^{n} & = & \pm\alpha ux^{n} & & \overset{+1}{\nu} = \pm\frac{1}{n+1}\alpha \\ '' & & '' & & '' \\ '' & & '' & & '' \end{array}$$

His valoribus coefficientium repertis, iterum inquirendum eſt, num in forma feriei

$$\log.(1+z)=\alpha z+\beta z^2+\gamma z^3+\delta z^4\ldots\ldots$$

tributa quidquam abſurdi lateat, nec ne. Productum $u^{m}x^{n-m+1}$ in priori evolutione coefficientem habet

$$\pm\mu^{n}\overset{-m+1}{\mathfrak{N}}=\pm\frac{1}{m}\alpha.\frac{n.(n-1)..(n-n+m)}{1.2..(n-m+1)}\left[\frac{(m-1)..(m-2)...(n-m+2)}{(n-m+2)..(n-m+3)...(m-1)}\right]=\pm\frac{1}{m}\alpha^{n}\overset{-1}{\mathfrak{M}}$$

in poſteriori:

$$\pm\overset{+1}{\nu}{}^{n+1}\mathfrak{M}=\pm\frac{\alpha}{n+1}\frac{(n+1).n.(n-1)..(n-m+2)}{1.2.3.....m}=\pm\frac{\alpha}{m}\frac{n.(n-1)..(n-m+2)}{1.2..(m-1)}=\pm\frac{\alpha}{m}{}^{n}\overset{-1}{\mathfrak{M}}$$

Supereſt, ut oſtendatur, utrumque coefficientem ſimul eſſe aut poſitivum, aut negativum pro paritate vel imparitate numeri n.

Sit enim n par, m impar, erit

$\overset{+1}{\nu}=+\frac{1}{n+1}\alpha,\ \mu=+\frac{1}{m}\alpha$ et $(n-m+1)$ numerus par, habebit igitur ${}^{n}\overset{-m+1}{\mathfrak{N}}$ ſignum $+$.

Sit n impar, m impar, erit

$\overset{+1}{\nu}=-\frac{1}{n+1}\alpha,\ \mu=+\frac{1}{m}\alpha$ et $(n-m+1)$ numerus impar, habebit igitur ${}^{n}\overset{-m+1}{\mathfrak{N}}$ ſignum $-$

reliqua.

§. 5.

§. 5.

THEOREMA.

In duabus §§. praecedentibus logarithmicarum ſerierum

$$p^x = 1 + \frac{Bx}{1} + \frac{Bx^2}{1.2} + \ldots + \frac{Bx^n}{1.2\ldots n} \ldots \text{et } \log.(1+z) = \alpha[z - \tfrac{1}{2}z^2 + \tfrac{1}{3}z^3 - \tfrac{1}{4}z^4 \ldots \pm \frac{z^n}{n} \ldots]$$

non alteram ab altera derivavimus, ſed quamlibet ſeparatim adduximus. Oſtendendum igitur eſt, qualis nexus ſit inter B, α et p in ſingulo quoque logarithmorum ſyſtemate.

Ponamus $x = \log.(1+z) = \alpha[z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4 + \frac{1}{5}z^5 \ldots]$ ſecundum §. 3. erit $(1+z) = p^x = p^{\log.(1+z)} = 1 + B\log.(1+z) + B^2\frac{\log.(1+z)^2}{1.2} + \ldots$

$$= 1 + B\alpha[z - \tfrac{1}{2}z^2 + \ldots] + \frac{B^2\alpha^2}{1.2}.[z - \tfrac{1}{2}z^2 + \ldots]^2 + \ldots$$

et terminis ad potentias variabilis z revocatis,

$$1 + z = 1 + B\alpha z - \left.\begin{matrix}\frac{B\alpha}{2} \\ + \frac{B^2\alpha^2}{1.2}\end{matrix}\right| \begin{matrix} z^2 \ldots \\ \ldots \end{matrix}$$

Ex qua aequatione patet, ſemper eſſe $B\alpha = 1$ et Log. $\alpha = -$ Log. B, cumque ſit in ſyſtemate naturalium logarithmorum $\alpha = 1$, ſimul eſſe $B = 1$. Reperitur itaque naturalis ſyſtematis baſis h, poſitis h, 1, pro p et B

$$h = 1 + \tfrac{1}{1} + \tfrac{1}{1.2} + \tfrac{1}{1.2.3} + \ldots = 2,718\,281\,828\,4 \ldots$$

in ſyſtemate quocunque vero, cuius baſin exprimit p, eſt

$$1 = \log.p = \alpha \log.\text{nat.}\,p \text{ ideoque } \alpha = \frac{1}{\log.\text{nat.}\,p}$$

et quia $B = \frac{1}{\alpha}$, $B = \log.$ nat. p,

nec non propter $p^x = 1 + \frac{Bx}{1} + \frac{B^2x^2}{1.2} + \frac{B^3x^3}{1.2.3} \ldots$

poſito $x = 1$ $\quad p = 1 + \log.\text{nat.}\,p + \frac{(\log.\text{nat.}\,p)^2}{1.2} + \frac{(\log.\text{nat.}\,p)^3}{1.2.3} + \ldots$

§. 6.

Cum noſtri non fuerit conſilii in hoc contexendo libello, primariis logarithmicis ſeriebus adductis, logarithmis ipſis diutius immorari, pergimus ad eas ſeries, quibus ſinus et coſinus per arcus ſuos exprimuntur, quod earum evolutio difficultatibus quibusdam pariter laborat, in eisque reliquae trigonometricae ſeries, veluti in fundamento, nituntur.

§. 7.

THEOREMA.

Posito radio = 1, sinus et cosinus his seriebus exprimuntur

$$\text{Sin. } z = z - \frac{z^3}{1.2.3} + \frac{z^5}{1.2.3.4.5} - \frac{z^7}{1.2.3.4.5.6.7} + \ldots \pm \frac{z^{2n-1}}{1.2\ldots(2n-1)} \mp \ldots\ldots$$

$$\text{Cos. } z = 1 - \frac{z^2}{1.2} + \frac{z^4}{1.2.3.4} - \frac{z^6}{1.2.3.4.5.6} + \frac{z^8}{1.2.3.4.5.6.7.8} - \ldots \mp \frac{z^{2n}}{1.2..2n} \pm \ldots$$

DEMONSTRATIO.

I. Ponamus esse $\text{Cos. } z = 1 + Bz^2 + Dz^4 + Fz^6 + \ldots + \overset{+n}{N}z^{2n} \ldots$

erit pro $z = x+u$, $\text{Cos.}(x+u) = 1 + B(x+u)^2 + D(x+u)^4 + F(x+u)^6 + \ldots + \overset{+n}{N}(x+u)^{2n} \ldots$

pro $z = x-u$, $\text{Cos.}(x-u) = 1 + B(x-u)^2 + D(x-u)^4 + F(x-u)^6 + \ldots + \overset{+n}{N}(x-u)^{2n} \ldots$

pro $z = x$, $\text{Cos. } x = 1 + Bx^2 + Dx^4 + Fx^6 + \ldots + \overset{+n}{N}x^{2n} \ldots$

pro $z = u$, $\text{Cos. } u = 1 + Bu^2 + Du^4 + Fu^6 + \ldots + \overset{+n}{N}u^{2n} \ldots$

Quod vero $\text{Cos.}(x+u) + \text{Cos.}(x-u) = 2\,\text{Cos. } x\,\text{Cos. } u$, ex substitutione illarum serierum in hac aequatione sequitur

$$(1+B(x+u)^2+D(x+u)^4+..)+(1+B(x-u)^2+D(x-u)^4+..) = 2.(1+Bx^2+Dx^4..).(1+Bu^2+Du^4..)$$

Productum ad dextram signi aequalitatis factoribus constat, qui imparium exponentium potentias variabilium x et u excludunt, eamque ob caussam ab ipso producto, ubi evolutum est, exulant, et termini, qui, evolutis binomiorum $x+u$, $x-u$ potentiis ad sinistram signi aequalitatis, impares variabilium x, u potentias continent, ut sese destruant, necesse est, ac propterea in ipsa statim evolutione omitti possunt. Hinc sequitur, reductione facta:

$$1 + B\left|\begin{matrix} x^2 \\ u^2 \end{matrix}\right. + D\left|\begin{matrix} x^4 \\ 6u^2x^2 \\ u^4 \end{matrix}\right. + F\left|\begin{matrix} x^6 \\ 15u^2x^4 \\ 15u^4x^2 \\ u^6 \end{matrix}\right. .. + \overset{n}{N}\left|\begin{matrix} x^{2n} \\ {}^{2n}\mathfrak{B}u^2x^{2n-2} \\ {}^{2n}\mathfrak{D}u^4x^{2n-4} \\ {}^{2n}\mathfrak{F}u^6x^{2n-6} \\ {}'' \\ {}^{2n}\overset{+m}{\mathfrak{M}}u^{2m}x^{2n-2m} \\ {}'' \\ {}^{2n}\overset{+n}{\mathfrak{N}}u^{2n} \end{matrix}\right. \ldots = \begin{matrix} 1 + Bx^2 + Dx^4 + Fx^6 .. + \overset{n}{N}x^{2n} \ldots \\ Bu^2 + BBu^2x^2 + BDu^2x^4 .. + B\overset{n-2}{N}u^2x^{2n-2} .. \\ Du^4 + DBu^4x^2 .. + D\overset{n-4}{N}u^4x^{2n-4} .. \\ + Fu^6 .. + F\overset{n-6}{N}u^6x^{2n-6} .. \\ '' \\ + \overset{m}{M}\overset{n-2m}{N}u^{2m}x^{2n-2m} .. \\ '' \\ + \overset{n}{N}u^{2n} \ldots \end{matrix}$$

et

et ex comparatione linearum secundarum secundum §. 3. Coroll. instituta

$$\frac{1.2}{1.2}Bu^2 = Bu^2 \quad \text{ergo} \quad B=\frac{2B}{1.2}$$

$$\frac{4.3}{1.2}Du^2x^2 = BBu^2x^2 \qquad D=\frac{2^2B^2}{(4.3)2}$$

$$\frac{6.5}{1.2}Fu^2x^4 = BDu^2x^4 \qquad F=\frac{2^3B^3}{(6.5).4.3.2}$$

$$\frac{8.7}{1.2}Hu^2x^6 = BFu^2x^6 \qquad H=\frac{2^4B^4}{(8.7).6.5.4.3.2}$$

″ ″ ″ ″

″ ″ ″ ″

$$^{2n}\mathfrak{B}\overset{n}{N}u^2x^{2n-2} = B\overset{n-2}{N}u^2x^{2n-2} \qquad \overset{n}{N}=\frac{2^nB^n}{2n(2n-1).[(2n-2)\dots 3.2]}$$

Superest, ut cognoscamus, num possit adoptata forma seriei

$$\text{Cos.}z=1+Bz^2+Dz^4\dots+\overset{n}{N}z^{2n}\dots$$

admitti. Quod vero in evolutione ad sinistram coefficiens producti variabilium $u^{2m}x^{2n-2m}$ est

$$\overset{n}{N}\,{}^{2n}\overset{m}{\mathfrak{M}}=\frac{2^nB^n}{2n(2n-1)\dots(2n-2m+1).(2n-2m)\dots 3.2}\cdot\frac{2n(2n-1)\dots(2n-2m+1)}{1.2\dots 2m}=\frac{2^nB^n}{(2n-2m)\dots 3.2.\,1.2\dots 2m}$$

in evolutione ad dextram

$$\overset{m}{M}\,\overset{n-2m}{N}=\frac{2^mB^m}{2m.(2m-1)\dots 3.2}\cdot\frac{2^{n-m}B^{n-m}}{(2n-2m)\dots 3.2}=\frac{2^nB^n}{2m.(2m-1)\dots 3.2.(2n-2m).(2n-2m-1)\dots 3.2}$$

ex horum coefficientium aequalitate, adoptatam formam sibi constare, ideoque admittendam esse, liquet, et habemus:

$$\text{Cos.}z=1+\frac{2Bz^2}{1.2}+\frac{2^2B^2z^4}{1.2.3.4}+\frac{2^3B^3z^6}{1.2.3.4.5.6}+\frac{2^4B^4z^8}{1.2.3.4.5.6.7.8}\dots+\frac{2^nB^nz^{2n}}{1.2\dots 2n}\dots$$

II. E serie, quae Cosinum exprimit, excludi vidimus ipsius arcus potentias imparium exponentium, itaque non, uti supra, ex hypothesi, sed nunc iure ponitur:

$$\text{Cos.}(x-u)=1+B(x-u)^2+D(x-u)^4\dots \quad \text{et} \quad \text{Cos.}(x+u)=1+B(x+u)^2+D(x+u)^4\dots$$

et binomiorum potentiis evolutis, terminisque, qui se mutuo tollunt, deletis, hinc provenit:

$$\frac{\text{Cos.}(x-u)-(\text{Cos.}(x+u)}{2}=-\Big[B2u^1x+D\left|\begin{matrix}4.u^1x^3\\4.u^3x\end{matrix}\right.+F\left|\begin{matrix}6.u^1x^5\\20.u^3x^3\\6.u^5x\end{matrix}\right.+H\left|\begin{matrix}8.u^1x^7\\56.u^3x^5\\56.u^5x^3\\8.u^7x\end{matrix}\right.\dots+\overset{+n}{N}\left|\begin{matrix}{}^{2n}\mathfrak{A}u^1x^{2n-1}\dots\\{}^{2n}\mathfrak{C}u^3x^{2n-3}\\{}^{2n}\mathfrak{E}u^5x^{2n-5}\\{}^{2n}\mathfrak{G}u^7x^{2n-7}\\″\\″\\{}^{2n}\overset{m-1}{\mathfrak{M}}u^{2m-1}x^{2n-2m+1}\\″\\″\\{}^{2n}\overset{n-1}{\mathfrak{N}}u^{2n-1}x\end{matrix}\right.\Big]$$

Iam

Iam fingamus $\sin. x = \alpha x + \gamma x^3 + \varepsilon x^5 + \eta x^7 \ldots + \overset{n-1}{\nu} x^{2n-1} \ldots$

$\sin. u = \alpha u + \gamma u^3 + \varepsilon u^5 + \eta u^7 \ldots + \overset{n-1}{\nu} u^{2n-1} \ldots,$

multiplicatione ambarum ferierum prodibit:

$$
\begin{aligned}
\text{Sin.}\, x\, \text{Sin.}\, u = \alpha\alpha u x + \alpha\gamma u^1 x^3 + \alpha\varepsilon u^1 x^5 + \alpha\eta u^1 x^7 \ldots & \alpha \overset{n-1}{\nu} u^1 x^{2n-1} \ldots \\
\gamma\alpha u^3 x^1 + \gamma\varepsilon u^3 x^3 + \gamma\varepsilon u^3 x^5 \ldots & \gamma \overset{n-3}{\nu} u^3 x^{2n-3} \\
\varepsilon\alpha u^5 x^1 + \varepsilon\gamma u^5 x^3 \ldots & \varepsilon \overset{n-5}{\nu} u^5 x^{2n-5} \\
\eta\alpha u^7 x^1 \ldots & \eta \overset{n-7}{\nu} u^7 x^{2n-7} \\
& \quad '' \\
& \quad '' \\
& \overset{m-1}{\mu} \overset{n-2m+1}{\nu} u^{2m-1} x^{2n-2m+1} \\
& \quad '' \\
& \quad '' \\
& \overset{n-1}{\nu} \alpha u^{2n-1} x^1
\end{aligned}
$$

Quod vero $\text{Sin.}\, x\, \text{Sin.}\, u = \frac{\text{Cof.}(x-u) - \text{Cof.}(x+u)}{2}$ in evolutionibus harum functionum terminos eadem variabilium producta complectentes fecundum §. 3. Coroll. oportet effe aequales, linearum horizontalium primarum comparatione invenitur:

$$
\begin{array}{llll}
\alpha\alpha u x = -2Bux & \text{itaque } \alpha = -\frac{2B}{\alpha} & \text{hoc eft fecundum I.} & \alpha = -\frac{2B}{\alpha} \\
\alpha\gamma u x^3 = -4Dux^3, & \gamma = -\frac{4D}{\alpha} & & \gamma = -\frac{2^2 B^2}{3.2\,\alpha} \\
\alpha\varepsilon u x^5 = -6Fux^5, & \varepsilon = -\frac{6F}{\alpha} & & \varepsilon = -\frac{2^3 B^3}{5.4.3.2.\alpha} \\
\alpha\eta u x^7 = -8Hux^7 & \eta = -\frac{8H}{\alpha} & & \eta = -\frac{2^4 B^4}{7.6.5.4.3.2.\alpha} \\
\quad '' \qquad '' & \quad '' \qquad '' & & \qquad '' \\
\quad '' \qquad '' & \quad '' \qquad '' & & \qquad '' \\
\alpha \overset{n-1}{\nu} u x^{2n-1} = -2n\overset{n}{N} u x^{2n-1} & \overset{n-1}{\nu} = -\frac{2n\overset{n}{N}}{\alpha} & & \overset{n-1}{\nu} = -\frac{2^n B^n}{(2n-1).(2n-2)\ldots 3.2.\alpha}
\end{array}
$$

Supereft igitur, ut doceamus, eam, quam feriebus $=$ Sin. x et Sin. u tribuimus, formam fibi conftare nullosque adducere coefficientium $\alpha, \gamma, \varepsilon, \eta \ldots \overset{n-1}{\nu} \ldots$ valores, qui inter fe pugnent. In evolutione functionis $\frac{\text{Cof.}(x-u) - \text{Cof.}(x+u)}{2}$ productum variabilium $u^{2m-1} x^{2n-2m+1}$ coefficientem habet:

$-\overset{n}{N}$

$$-\overset{n}{N}{}^{2n}\overset{m-1}{\mathfrak{M}} = \frac{-2^n B}{2n.(2n-1)..(2n-2m+2)..3.2} \cdot \frac{2n.(2n-1)..(2n-2m+2)}{1.2...2m-1} = \frac{-2^n B^n}{(2n-2m+1)..3.2;2.3..(2m-1)}$$

in evolutione vero functionis Sin. x Sin. u:

$$\overset{m-1}{\mu}\ \overset{n-2m+1}{\nu} = \frac{-2^m B^m}{(2m-1)..3.2.\alpha} \cdot \frac{-2^{n-m+1} B^{n-m+1}}{(2n-2m+1)..3.2.\alpha} = \frac{2^n \cdot B^n}{(2m-1)..3.2.(2n-2m+1)..3.2} \cdot \frac{2B}{\alpha^2},$$

qui ad prioris coefficientis valorem redit, quod fupra $\frac{2B}{\alpha^2} = -1$ reperimus.

Hinc fequitur:

$$\text{Sin.} z = -\left[\frac{2Bz}{\alpha} + \frac{2^2B^2z^3}{3.2.\alpha} + \frac{2^3B^3z^5}{5.4.3.2.\alpha} + \frac{2^4B^4z^7}{7.6.5.4.3.2.\alpha} \ldots + \frac{2^nB^nz^{2n-1}}{(2n-1).(2n-2)..3.2\alpha} \ldots\right]$$

five, quod eft $-2B = +\alpha^2$,

$$\text{Sin.} z = \frac{\alpha z}{1} - \frac{\alpha^3z^3}{1.2.3} + \frac{\alpha^5z^5}{1.2.3.4.5} - \frac{\alpha^7z^7}{1.2.3.4.5.6.7} \ldots \pm \frac{\alpha^{2n-1}z^{2n-1}}{1.2.3\ldots(2n-1)} \pm \ldots$$

et eandem ob caufam:

$$\text{Cof.} z = 1 - \frac{\alpha^2z^2}{1.2} + \frac{\alpha^4z^4}{1.2.3.4} - \frac{\alpha^6z^6}{1.2.3.4.5.6} + \frac{\alpha^8z^8}{1.2.3.4.5.6.7.8} \ldots \frac{\mp\alpha^{2n}z^{2n}}{1.2.3\ldots 2n} \pm \ldots$$

III. In utraque ferie quod ubique eosdem fimul habent exponentes α et z nulla comparatione terminorum five coefficientium relatione litterae α valor eruitur, quippe ex aequationibus cum z fimul tollitur divifione. Aliter itaque rem adgreffus eft LAGRANGIUS (*Theorie des fonctions analytiques*, §. 28), quem nunc fequimur.

Eft Tang $z > z$, ergo $\frac{\text{Sin.} z}{\sqrt{1-\text{Sin.}^2 z}} > z$ fed etiam Sin. $z < z$, ergo $1 > \frac{\text{Sin.} z}{z}$

$$\text{Sin.} z > z\sqrt{1-\text{Sin.}^2 z} \qquad \text{et propter (1)}\ 1 > \sqrt{\frac{1}{1+z^2}}$$

$$\text{Sin.}^2 z(1+z^2) > z^2 \qquad \frac{\sqrt{1+z^2}}{1+z} > \frac{1}{1+z^2}$$

$$1.)\ \frac{\text{Sin.} z}{z} > \sqrt{\frac{1}{1+z^2}} \qquad 2.)\ \sqrt{\frac{1}{1+z^2}} > \frac{1}{1+z^2}$$

Quod $\frac{1}{1+z^2} - (1-z^2) = \frac{z^4}{1+z^2}$, eft $\frac{1}{1+z^2} > 1 - z^2$, ergo propter (1) et (2)

3.) $\frac{\text{Sin.} z}{z} > 1 - z^2$ et ut paullo ante inveneramus 4.) $\frac{\text{Sin.} z}{z} < 1$

Iam

Iam ponamus arcui z talem dari valorem, ut sit $\frac{\alpha^3 z^2}{1.2.3} < \alpha$, id quod accidit, quando $z < \frac{\sqrt{6}}{\alpha}$, tum series: $\frac{\text{Sin.}\, z}{z} = \alpha - \frac{\alpha^3 z^2}{1.2.3} + \frac{\alpha^5 z^4}{1.2.3.4.5} - \frac{\alpha^7 z^6}{1.2.3.4.5.6.7} + \ldots$ converget *) eamque ob rem terminorum $\frac{\alpha^5 z^4}{1.2.3.4.5} - \frac{\alpha^7 z^6}{1.2.3.4.5.6.7} + \ldots$ summa erit positiva, quam indicabimus per literam P. Quo facto, $\frac{\text{Sin.}\, z}{z} = \alpha - \frac{\alpha^3 z^2}{1.2.3} + P$

et tam $\alpha - \frac{\text{Sin.}\, z}{z} = \frac{\alpha^3 z^2}{1.2.3} - P$
quam $\frac{\text{Sin.}\, z}{z} - \alpha + \frac{\alpha^3 z^2}{1.2.3} = P$ } quantitas erit positiva propter convergentiam seriei

ideoque $\alpha > \frac{\text{Sin.}\, z}{z}$ et $\alpha - \frac{\alpha^3 z^2}{1.2.3} < \frac{\text{Sin.}\, z}{z}$, nec non propter (3) et (4)

(5.) $\alpha > 1 - z^2$ et $\alpha - \alpha^3 z^2 < 1$ unde sequitur (6.) $\alpha < 1 + \frac{\alpha^3 z^2}{1.2.3}$.

His patet, quum dato quocunque valore minor esse possit arcus z, inaequalitates (5) et (6) non semper obtinere posse, nisi sit $\alpha = 1$. Posito enim $\alpha < 1$, tantillus esse potest z, ut $1 - z^2 > \alpha$, quod repugnat inaequalitati V. $1 - z^2 < \alpha$. Contra, posito $\alpha > 1$, propter arcus z parvitatem fieri potest, ut differentiam $\alpha - 1$ non aequet $\frac{\alpha^3 z^2}{1.2.3}$, et sit $\alpha - 1 > \frac{\alpha^3 z^2}{1.2.3}$, sive $\alpha > 1 + \frac{\alpha^3 z^2}{1.2.3}$, id quod inaequalitati (6) $\alpha < 1 + \frac{\alpha^3 z^2}{1.2.3}$ contrarium est.

*) Quotiescunque in serie huius formae $A - B + C - D + E - F + \ldots$

I. Summae terminorum binorum proxime sese excipientium, si incipias a secundo et tertio, sint negativae, e. g. $-B + C$, $-D + E$, $-F + G$, ...

II. Summae terminorum binorum proxime sese excipientium, si incipias a tertio et quarto, sint positivae, e. g. $C - D$, $E - F$, $G - H$, ...

seriem ipsam convergere, Lagrangius contendit l. c., quod hoc modo ostendi potest. Sit enim omnium summarum illarum negativarum valor $= -R$ et positivarum $= + S$, erit series ipsa $A - B + C - D + \ldots = A - R = A - B + S$, ideoque $R = + B - S$, ergo terminus secundus B maior quam summa S reliquorum. Iisdem argumentis reperitur, quemque terminum terminorum insequentium omnium summam excedere q. e. d. Haec vero convergentiae indicia in serie $= \frac{\text{Sin.}\, z}{z}$ deprehendi, palam est.

IV.

THEOREMATIS BINOMIALIS

DEMONSTRATIO ELEMENTARIS.

THEOREMATIS BINOMIALIS
DEMONSTRATIO ELEMENTARIS.

§. 1.

Demonſtrationem, quam Pascalius et Neutonus theoremati binomiali addendam poſteris reliquerant, plures tentarunt, eaque re demonſtrationum huius theorematis duo exorta ſunt genera, quorum alterum differentialium requirit opem, alterum demonſtrationes eas complectitur, quas elementares vocari placuit, quippe intra analyſeos finitorum terminos ſubſiſtunt. Sed quoniam Kaestneriana demonſtratio omnesque, quae primum conſtituunt genus, non ſine magno analyſeos totius incommodo admittuntur, quippe hoc theoremate ad analyſin infinitorum reiecto, analyſi finitorum nimium eripitur, multaque, quae iuncta eſſe deberent, divelluntur ſedibusque ſuis pelluntur, quae demonſtrationis elementaris auxilio analyſi finitorum reddi, locisque ſuis reponi poſſunt; noſtris temporibus plures videmus elementares proferri demonſtrationes, praeſtantia vero ſua inter ſe diſſimiles. Alii enim ſibi ipſis illuſerunt analogiis, quae ut ad res novas inveniendas plurimum faciunt, ſic in demonſtrandis Theorematibus iam repertis improbiſſime fallunt, alii ſingula exponentium genera diligenter perſequi coacti ſunt, nec tamen ad omnes exponentium valores, ſi vera fateamur, demonſtrationes ſuas protulerunt. Quarum demonſtrationum numero utrum noſtra addenda, an eximenda ſit, non noſtri, ſed aliorum iudicii eſt.

§. 2.

EXPLICATIO.

Variationes elementorum α, β, γ .... ad ſummam n complexiones combinatorias continent omnes, in quarum qualibet numeri 1, 2, 3....

 elemen-

elementorum literalium loco positi sua ipsorum summa numerum n constanter aequant, sic $\alpha\alpha\gamma$, $\alpha\beta\beta$, $\alpha\gamma\alpha$, $\beta\alpha\beta$, $\beta\beta\alpha$, $\gamma\alpha\alpha$, Variationes sunt elementorum α, β, γ.... ad summam 5, quod substitutis numeris 1, 2, 3 .... literarum loco, prodeunt complexiones numericae

113, 122, 131, 212, 221, 311,

in quibus singulis elementorum summa est 5. Caeterum complexiones illae literales, quod quaelibet tribus literis conflata est et praeter adpositas nulla datur, tertiam Variationum classem ad summam 5 constituunt et hoc signo $\underset{(\alpha,\beta,\gamma\ldots)}{{}^{5}C}$ notantur, ubi litera obliqua C tertiam classem Variationum, index subiectus elementa, exponens ad laevam adpositus summam indicit, ad quam complexiones referantur.

§. 3.

PROBLEMA.

Variationum Classem quamlibet ex datis elementis ad summam definitam construere.

SOLVTIO.

Antequam generalem problematis solutionem sumus daturi, Classem Variationum quartam ex elementis α, β, γ, δ... ad summam 7 adponemus, ut facilius, exemplo praemisso, generaliores regulae capiantur.

$$\underset{(\alpha,\beta,\gamma,\delta\ldots)}{{}^{7}D} = \begin{matrix} \alpha\,\alpha\,\alpha\,\delta & \beta\,\alpha\,\alpha\,\gamma & \gamma\,\alpha\,\alpha\,\beta & \delta\,\alpha\,\alpha\,\alpha \\ \alpha\,\alpha\,\beta\,\gamma & \beta\,\alpha\,\beta\,\beta & \gamma\,\alpha\,\beta\,\alpha & \\ \alpha\,\alpha\,\gamma\,\beta & \beta\,\alpha\,\gamma\,\alpha & \gamma\,\beta\,\alpha\,\alpha & \\ \alpha\,\alpha\,\delta\,\alpha & \beta\,\beta\,\alpha\,\beta & & \\ \alpha\,\beta\,\alpha\,\gamma & \beta\,\beta\,\beta\,\alpha & & \\ \alpha\,\beta\,\beta\,\beta & \beta\,\gamma\,\alpha\,\alpha & & \\ \alpha\,\beta\,\gamma\,\alpha & & & \\ \alpha\,\gamma\,\alpha\,\beta & & & \\ \alpha\,\gamma\,\beta\,\alpha & & & \\ \alpha\,\delta\,\alpha\,\alpha & & & \end{matrix}$$

1) Prima quidem Variationum classis $(m)^{tae}$ ad summam n reperitur sedibus omnibus praeter novissimam ad dextram litera α expletis, novissima vero ea litera, quae numero $(n-m+1)$ respondet.

2) Proxime sequentes Variationes hac lege reperiuntur, ut non mutatis literis, quae priores sedes tenent, in penultima sede sese excipiant elementa

menta α, β, γ . . . ex ordine, in ultima vero ordine inverso, donec ultimam sedem occupet ipsa litera α.

3) Tum in tertia sede seu antepenultima β ponitur loco α, in penultima α, in ultima ea litera, quae numero $n-m$ respondet, et proxime sequentes complexiones secundum regulam 2 adduntur.

4) Literae γ, δ, ε . . . . pariter in tertium locum migrant, et adductis omnibus Variationibus, quae tantum tribus postremis ad dextram literis inter se discrepant, mutatio elementorum ad quartam, quintam reliquasque sedes pergit, donec perveniatur ad complexionem, quae primam refert, sed ordine literarum inverso.

Vix monendum esse putavimus lectorem, in exemplo Variationum, quod his regulis praemisimus, esse $n=7$, $m=4$, neque severam demonstrationem addidimus huic solutioni, res enim loquitur ipsa.

§. 4.

PROBLEMA.

Variationes ad summam datam ex elementis α, β, γ . . . . ita construere omnes, ut minorum summarum Variationes expedite possint seiungi vel execari.

SOLVTIO.

α	α	α	α
	β	α	α
	α	β	α
		γ	α
	α	α	β
		β	β
		α	γ
			δ

Adposita Variationum ad summam 4 tabula constructa est et continuari potest secundum regulas sequentes.

Variationibus ad summam n cruribus anguli recti insertis, extra angulum adponatur ad dextram α, hisque subscribantur eiusdem summae n Variationes, hac tamen lege, ut novissimo quoque elemento in proxime sequens converso, ipsae producant Variationes summae $(n+1)$, quo facto nulla Variationum ad summam $(n+1)$ deerit.

DEMONSTRATIO.

Si quis omnes Variationes ad summam datam sic posse obtineri neget et e. g. inter Variationes ad summam 8 non deprehendi contendat,

$$\gamma\beta\gamma \quad \text{et} \quad \beta\alpha\alpha\beta\beta,$$

is simul in Variationibus nostris

ad

ad summam 7 adesse Variationem γββ et βααβα
6 . . . . . . γβα . βααβ
5 . . . . . . γβ . βααα
4 . . . . . . γα . βαα
3 . . . . . . γ . βα
2 . . . . β

negabit, id quod foret absurdum.

§. 5.

EXPLICATIO.

Complexionum constructio quaelibet, quae sive classis, sive summae, sive indicis cuiusdam complexiones ita disponit, ut sive classium, sive summarum, sive indicum priorum complexiones rite, eademque lege dispositas simul exhibeat, *Involutio* vocatur. Sic in §. 4. Involutionem Variationum ad summam 4 proposuimus, et regulas dedimus, quibus Variationum Involutiones ad summas quascunque producantur. Involutiones autem Variationum ex elementis α, β, γ, ... ad summam *n*, peculiari signo $\underset{(\alpha,\beta,\gamma..\nu)}{{}^{n}\mathfrak{J}}$ et si singulis Variationibus ad sinistram numeri integri primis elementis respondentes praefigantur, signo $\underset{(\alpha,\beta,\gamma..\nu)}{\mathrm{j}\,{}^{n}\mathfrak{J}}$ notabuntur.

EXEMPLUM I.

α	α	α	α	α
	β	α	α	α
	α	β	α	α
		γ	α	α
	α	α	β	α
		β	β	α
		α	γ	α
			δ	α
	α	α	α	β
		β	α	β
		α	β	β
			γ	β
		α	α	γ
			β	γ
			α	δ
				ε

$$= \underset{(\alpha,\beta,\gamma,\delta,\varepsilon)}{{}^{5}\mathfrak{J}}$$

EXEMPLUM II.

1.α	α	α	α	α
	2.β	α	α	α
	1.α	β	α	α
		3.γ	α	α
	1.α	α	β	α
		2.β	β	α
		1.α	γ	α
			4.δ	α
	1.α	α	α	β
		2.β	α	β
		1.α	β	β
			3.γ	β
		1.α	α	γ
			2.β	γ
			1.α	δ
				5.ε

$$= \underset{(\alpha,\beta,\gamma,\delta,\varepsilon)}{\mathrm{j}\,{}^{5}\mathfrak{J}}$$

§. 6.

§. 6.

THEOREMA.

Est $\underset{(\alpha,\beta,\gamma..\nu)}{j^n\mathfrak{J}} = \underset{(\alpha,\ \beta,\ \gamma,\ \delta\ \ldots\ \overset{-1}{\nu})}{j^{n-1}\mathfrak{J}\alpha + j^{n-2}\mathfrak{J}\beta + j^{n-3}\mathfrak{J}\gamma + j^{n-4}\mathfrak{J}\delta \ldots j^{n-m}\mathfrak{J}\mu + \ldots j^2\mathfrak{J}\overset{-2}{\nu} + j^1\mathfrak{J}\overset{-1}{\nu} + n\nu}$,

cuius theorematis ratio ex ipſo involutionum modo propoſitarum adſpectu cognoſcitur.

§. 7.

THEOREMA.

Eſt Log.nat.$[1-(\alpha x+\beta x^2+\gamma x^3..)] = -\underset{(\alpha,\ \beta,\ \gamma,\ \delta\ \ldots)}{[j^1\mathfrak{J}\frac{x}{1}+j^2\mathfrak{J}\frac{x^2}{2}+j^3\mathfrak{J}\frac{x^3}{3}+j^4\mathfrak{J}\frac{x^4}{4}\ldots+j^n\mathfrak{J}\frac{x^n}{n}\ldots]}$

ſive, Variationibus ipſis ſignorum involutoriorum loco poſitis,

$$= -\left[1.\alpha\frac{x}{1} + \left.\begin{array}{r} 1.\alpha\ \alpha \\ 2.\beta \end{array}\right|\frac{x^2}{2} + \left.\begin{array}{r} 1.\alpha\ \alpha\ \alpha \\ 2.\beta\ \alpha \\ 1.\alpha\ \beta \\ 3.\gamma \end{array}\right|\frac{x^3}{3} + \left.\begin{array}{r} 1.\alpha\ \alpha\ \alpha\ \alpha \\ 2.\beta\ \alpha\ \alpha \\ 1.\alpha\ \beta\ \alpha \\ 3.\gamma\ \alpha \\ 1.\alpha\ \alpha\ \beta \\ 2.\beta\ \beta \\ 1.\alpha\ \gamma \\ 4.\delta \end{array}\right|\frac{x^4}{4} + \ldots\right].$$

DEMONSTRATIO.

Conſtat eſſe Log. nat. $(1-z) = -[\frac{z^1}{1}+\frac{z^2}{2}+\frac{z^3}{3}+\frac{z^4}{4}+\frac{z^5}{5}+\ldots\frac{z^n}{n}\ldots]$ (Conferatur commentatio noſtra: *Functiones logarithmicae et trigonometricae in ſeries infinitas ſolutae*, §. 4.) et, poſito $z = \alpha x+\beta x^2+\gamma x^3+\ldots$, potentiae huius infinitinomii ſignis combinatoriis Variationum §. 2. explicatis expreſſae erunt:

$$\begin{array}{l} z = {}^1Ax + {}^2Ax^2 + {}^3Ax^3 + {}^4Ax^4 \ldots + {}^nAx^n \ldots \\ z^2 = {}^2Bx^2 + {}^3Bx^3 + {}^4Bx^4 + {}^5Bx^5 \ldots + {}^{n+1}Bx^{n+1} \ldots \\ z^3 = {}^3Cx^3 + {}^4Cx^4 + {}^5Cx^5 + {}^6Cx^6 \ldots + {}^{n+2}Cx^{n+2} \ldots \\ \quad " \qquad " \qquad " \qquad " \qquad " \qquad " \\ z^m = {}^mMx^m + {}^{m+1}Mx^{m+1} + {}^{m+2}Mx^{m+2} + {}^{m+3}Mx^{m+3} \ldots + {}^{n+m-1}Mx^{n+m-1} \ldots \\ \qquad (\alpha,\ \beta,\ \gamma,\ \delta\ \ldots) \end{array}$$

quarum

quarum ſubſtitutione in ſerie = log. nat. (1 — z) deprehendetur:

$$\mathrm{Log.n.}(1-z)=\mathrm{L.n.}[1-(\alpha x+\beta x^2..)]=-\Big[\tfrac{1}{1}{}^{1}Ax+\left.\begin{array}{l}\tfrac{1}{1}{}^{2}A\\ \tfrac{1}{2}{}^{2}B\end{array}\right|x^2+\left.\begin{array}{l}\tfrac{1}{1}{}^{3}A\\ \tfrac{1}{2}{}^{3}B\\ \tfrac{1}{3}{}^{3}C\end{array}\right|x^3+\left.\begin{array}{l}\tfrac{1}{1}{}^{4}A\\ \tfrac{1}{2}{}^{4}B\\ \tfrac{1}{3}{}^{4}C\\ \tfrac{1}{4}{}^{4}D\end{array}\right|x^4\ldots+\left.\begin{array}{l}\tfrac{1}{1}{}^{n}A\\ \tfrac{1}{2}{}^{n}B\\ \tfrac{1}{3}{}^{n}C\\ \tfrac{1}{4}{}^{n}D\\ \;''\\ \;''\\ \tfrac{1}{m}{}^{n}M\\ \;''\\ \;''\\ \tfrac{1}{n}{}^{n}N\end{array}\right|x^n..\Big]$$

$$(\alpha,\ \beta,\ \gamma,\ \delta\ldots)$$

Sive, levi mutatione facta:

$$\mathrm{Log.\,n.}[1-(\alpha x+\beta x^2+\gamma x^3..)]=-\Big[\tfrac{1}{1}{}^{1}A\left|\tfrac{x}{1}\right.+\left.\begin{array}{l}\tfrac{2}{1}{}^{2}A\\ \tfrac{2}{2}{}^{2}B\end{array}\right|\tfrac{x^2}{2}+\left.\begin{array}{l}\tfrac{3}{1}{}^{3}A\\ \tfrac{3}{2}{}^{3}B\\ \tfrac{3}{3}{}^{3}C\end{array}\right|\tfrac{x^3}{3}+\left.\begin{array}{l}\tfrac{4}{1}{}^{4}A\\ \tfrac{4}{2}{}^{4}B\\ \tfrac{4}{3}{}^{4}C\\ \tfrac{4}{4}{}^{4}D\end{array}\right|\tfrac{x^4}{4}..+\left.\begin{array}{l}\tfrac{n}{1}{}^{n}A\\ \tfrac{n}{2}{}^{n}B\\ \tfrac{n}{3}{}^{n}C\\ \tfrac{n}{4}{}^{n}D\\ \;''\\ \;''\\ \tfrac{n}{m}{}^{n}M\\ \;''\\ \;''\\ \tfrac{n}{n}{}^{n}N\end{array}\right|\tfrac{x^n}{n}\ldots\Big]$$

$$(\alpha,\ \beta,\ \gamma,\ \delta\ldots)$$

Omnis ergo Theorematis demonſtratio in eo vertitur, ut oſtendatur, ſummam claſſium

$$n\left[\frac{{}^{n}A}{1}+\frac{{}^{n}B}{2}+\frac{{}^{n}C}{3}\ldots+\frac{{}^{n}M}{m}\ldots+\frac{{}^{n}N}{n}\right]_{(\alpha,\ \beta,\ \gamma,\ \delta\ldots\nu)}$$ aequalem eſſe Involutioni $j^{n}\mathfrak{J}_{(\alpha,\beta,\gamma\ldots\nu)}$

Claſſes illae et haec Involutio Variationum easdem complexiones literales continent, diverſas tamen et ordine et coefficientibus ſuis. Vt igitur aequalitatem Claſſium et Involutionum demonſtremus, contemplemur exempli gratia complexiones literales has,

$\alpha\alpha\beta\delta$

α	α	β	δ	1	1	2	4
α	α	δ	β	1	1	4	2
α	β	α	δ	1	2	1	4
α	β	δ	α	1	2	4	1
α	δ	α	β	1	4	1	2
α	δ	β	α	1	4	2	1
β	α	α	δ	2	1	1	4
β	α	δ	α	2	1	4	1
β	δ	α	α	2	4	1	1
δ	α	α	β	4	1	1	2
δ	α	β	α	4	1	2	1
δ	β	α	α	4	2	1	1

quae referunt permutationes a complexione $\alpha\alpha\beta\delta$ oriundas numero 12, quarum singula quaeque Variationibus signo $\underset{(\alpha,\beta,\gamma\ldots)}{{}^8D}$ indictis inserta coefficientem habet $\frac{8}{4}$. Haec producta, ex complexionibus illis et communi omnium coefficiente $\frac{8}{4}$ ubi collecta sunt, summam producunt $= 12.\frac{8}{4}.\alpha\alpha\beta\delta$. Eaedem Permutationes in Involutione iunctae cernuntur numeris integris, quae ipsarum primis elementis respondent, atque sic instructae eandem summam efficiunt $= 12.\frac{8}{4}.\alpha\alpha\beta\delta$, id quod sequentia ratiocinia docebunt. Permutationibus scilicet literalibus adiectae numericae respondent, quadruplicem numerorum columnam exhibentes. Harum columnarum quaelibet eosdem numeros, sed alio ordine dispositos continet, id enim Permutationum poscit natura; singularum igitur columnarum summas oportet esse aequales, cumque sit omnium numerorum, quibus constant, summa 8. 12, uniuscuiusque columnae summa erit $= \frac{8.12}{4}$. Sed prima columna numeros exhibet, qui Permutationibus a producto $\alpha\alpha\beta\delta$ oriundis praefigantur in Involutione $\underset{(\alpha,\beta,\gamma\ldots)}{j^8\mathfrak{J}}$, itaque istae Permutationes in Involutione coefficientibus suis instructae pariter ad summam redeunt $= \frac{8.12}{4}.\alpha\alpha\beta\delta$. Eadem argumenta ad Permutationes elementorum cuiuslibet complexionis transferri posse patet, eaque re constat,

non esse modo $\underset{(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta,\vartheta)}{8\left[\frac{{}^8A}{1}+\frac{{}^8B}{2}+\frac{{}^8C}{3}+\frac{{}^8D}{4}+\frac{{}^8E}{5}+\frac{{}^8F}{6}+\frac{{}^8G}{7}+\frac{{}^8H}{8}\right]} = \underset{(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta,\vartheta)}{j^8\mathfrak{J}}$

sed etiam $\underset{(\alpha,\beta,\gamma,\delta\ldots\nu)}{n\left[\frac{{}^nA}{1}+\frac{{}^nB}{2}+\frac{{}^nC}{3}+\frac{{}^nD}{4}\ldots+\frac{{}^nM}{m}\ldots+\frac{{}^nN}{n}\right]} = \underset{(\alpha,\beta,\gamma,\delta\ldots\nu)}{j^n\mathfrak{J}}$ Q.E.D.*

§. 8.

THEOREMA.

Est $(1-x)^u = 1 - {}^u\mathfrak{A}x + {}^u\mathfrak{B}x^2 - {}^u\mathfrak{C}x^3 + {}^u\mathfrak{D}x^4 \ldots \pm {}^u\mathfrak{N}x^n \ldots$, ubi ${}^u\mathfrak{N} = \frac{u.(u-1).(u-2)\ldots(u-n+1)}{1.2.3\ldots n}$

DEMON-

* Idem Theorema paullo aliter proposuimus in libello: *Usus logarithmorum in theoria aequationum*, *Lipsiae 1796*. §. 11.

DEMONSTRATIO.

Secundum §. 4. commentationis nostrae supra citatae est:

$$\text{I. } u \text{ log. nat. } (1-x) = -u\left[\frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\ldots+\frac{x^n}{n}\ldots\right]$$

et in §. praecedenti invenimus

$$\text{II. log. nat. } [1-(\alpha x+\beta x^2..)] = -\left[j^1\frac{\mathfrak{J}x}{1}+j^2\frac{\mathfrak{J}x^2}{2}+j^3\frac{\mathfrak{J}x^3}{3}+j^4\frac{\mathfrak{J}x^4}{4}+\ldots+j^n\frac{\mathfrak{J}x^n}{n}\ldots\right]$$
$$(\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon\ldots.)$$

Iam ponamus esse

log. nat. $(1-x)^u = u$ log. nat. $(1-x)$ et $(1-x)^u = 1-(\alpha x+\beta x^2+\gamma x^3\ldots.)$, aequales erunt series I. et II. ex earumque comparatione prodibit

$$\text{III. } u = j^1\mathfrak{J} = j^2\mathfrak{J} = j^3\mathfrak{J}\ \ldots.. = j^n\mathfrak{J}$$
$$(\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon\ \ldots..\ \nu)$$

Sed quoniam §. 6. vidimus esse

$$\underset{(\alpha,\beta,..\nu)}{j^n\mathfrak{J}} = j^{n-1}\mathfrak{J}\alpha + j^{n-2}\mathfrak{J}\beta + j^{n-3}\mathfrak{J}\gamma\ldots+j^{n-m}\mathfrak{J}\mu\ldots+j^1\mathfrak{J}\overset{-1}{\nu}+n\nu,$$
$$(\alpha,\ \beta,\ \gamma,\ \delta,\ldots.\overset{-1}{\nu})$$

ex aequatione III. sequitur,

posito $n=1,\quad u = j^1\mathfrak{J} = 1\,\alpha$

$n=2,\quad u = j^2\mathfrak{J} = j^1\mathfrak{J}\alpha + 2\beta$

$n=3,\quad u = j^3\mathfrak{J} = j^2\mathfrak{J}\alpha + j^1\mathfrak{J}\beta + 3\gamma$

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$$n=n,\quad u = \underset{(\alpha,\beta,\gamma,..\nu)}{j^n\mathfrak{J}} = j^{n-1}\mathfrak{J}\alpha+j^{n-2}\mathfrak{J}\beta+j^{n-3}\mathfrak{J}\gamma\ldots+j^{n-m}\mathfrak{J}\mu\ldots+j^1\mathfrak{J}\overset{-1}{\nu}+n\nu,$$
$$(\alpha,\ \beta,\ \gamma,\ \delta,\ldots.\overset{-1}{\nu})$$

nec non, communi omnium harum Involutionum valore u earum loco substituto:

$$\begin{aligned}
u &= \alpha\\
u &= u\alpha + 2\beta\\
u &= u\alpha + u\beta + 3\gamma\\
u &= u\alpha + u\beta + u\gamma + 4\delta\\
u &\quad ″\quad ″\quad ″\quad ″\\
u &= u\alpha + u\beta + u\gamma + u\delta\ldots+u\overset{-2}{\nu}+(n-1)\overset{-1}{\nu}\\
u &= u\alpha + u\beta + u\gamma + u\delta\ldots+u\overset{-2}{\nu}+u\overset{-1}{\nu}+n\nu,
\end{aligned}$$

quarum

quarum aequationum qualibet de praecedente deducta, sequentes proveniunt differentiae:

		$\alpha = + u$	$= + {}^u\mathfrak{A}$
$o = (u-1)\alpha + 2\beta$	ex quibus, adiecto valore literae α, sequitur per substitutionem:	$\beta = - \frac{u.(u-1)}{1.2}$	$= - {}^u\mathfrak{B}$
$o = (u-2)\beta + 3\gamma$		$\gamma = + \frac{u.(u-1).(u-2)}{1.2.3}$	$= + {}^u\mathfrak{C}$
$o = (u-3)\gamma + 4\delta$		$\delta = - \frac{u.(u-1).(u-2).(u-3)}{1.2.3.4}$	$= - {}^u\mathfrak{D}$
"		"	"
$o = (u-n+1)\overset{-1}{\nu} + n\nu$		$\nu = \pm \frac{u.(u-1).(u-2).(u-3)..(u-n+1)}{1.2.3.4...n}$	$= \pm {}^u\mathfrak{N}$

Ergo est

$$(1-x)^u = 1 - {}^u\mathfrak{A}x + {}^u\mathfrak{B}x^2 - {}^u\mathfrak{C}x^3 + {}^u\mathfrak{D}x^4 - \ldots \pm {}^u\mathfrak{N}x^n \mp \ldots,$$

dum forma seriei $1 - (\alpha x + \beta x^2 + \gamma x^3 \ldots) = (1-x)^u$ tributa possit admitti. Huius rei investigandae causa ponamus, esse $(1-x)^u = 1-s$ et $s = \alpha x^a + \beta x^b + \gamma x^c ..$, erit:

$$\text{log. nat. } (1-x)^u = \text{log. nat. } (1-s) = - \left[\frac{s}{1} + \frac{s^2}{2} + \frac{s^3}{3} + \frac{s^4}{4} + \frac{s^5}{5} \ldots\right], \quad (P$$

Sed quoniam est etiam

$$\text{log. nat. } (1-x)^u = u \text{ log. nat. } (1-x) = - u\left[\frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} \ldots\right], \quad (Q$$

utriusque seriei P et Q comparatio docet, eam seriei s formam esse dandam, ut, substitutione seriei $\alpha x^a + \beta x^b + \gamma x^c \ldots$ in P facta, exponentes ab unitate incipiant et secundum differentiam 1 procedant, quippe formae serierum P et Q congruere debent, eamque ob rem exponentes a, b, c, ... aequare unitatem.

SCHOLION. Superest, ut ratio reddatur de valorum diversitate, quos exponens u admittit, sed hanc rem, quae pendet ab indole aequationis log. nat. $(1-x)^u = u$ log. nat. $(1-x)$, nec paucis absolvitur, ad sequentes §§. reiicimus.

§. 9.

Si quis forte ratiocinia nostra animo remissiori persequatur et nexum transmittat, qui huic cum praecedenti commentatione intercedit, facile ei in circulo circumagi videbimur; quia logarithmis ad demonstrandum theorema binomiale usi sumus et plerumque series logarithmicae ita proferun-

 tur,

tur, ut demonstratum iam esse theorema binomiale postulent. Sed in prioris commentationis §. 4. posito $z = - x$, et $\alpha = 1$, ad seriem

$$\text{log. nat. } (1 - x) = - \left(\frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} \ldots \right)$$

deducimur quidem binomialis theorematis ope, ita tamen, ut binomiis nulli nisi integri, tam positivi, quam negativi exponentes tribuantur, fracti vero non adhibeantur. Itaque ordinem, quo theoremata arctissimo vinculo iuncta commode sese excipiant, hunc esse arbitramur:

1) Binomium exponentis integri positivi,
2) Binomium exponentis integri negativi,
3) Polynomium exponentis integri positivi et negativi,
4) Logarithmus naturalis binomii $(1 \pm x)$,
5) Logarithmus naturalis polynomii $1 - (\alpha x + \beta x^2 + \gamma x^3 \ldots)$,
6) Binomium exponentis cuiuslibet n,
7) Polynomium exponentis eiusdem.

Demonstrationes primi et tertii Theorematis, quippe de eis inter omnes constat, non repetendae videntur, demonstratio quarti in priori commentatione continetur, quinti et sexti rationem modo reddidimus, septimum nominavimus tantum, ne desideraretur. Sed secundi Theorematis demonstrationem sumus daturi.

Ex divisione constat esse, omisso quidem residuo,

$$\text{I. } (1-x)^{-1} = \frac{1}{(1-x)} = 1 + 1x + 1x^2 + 1x^3 + 1x^4 + \ldots 1x^n \ldots = 1 - {}^{-1}\mathfrak{A}x + {}^{-1}\mathfrak{B}x^2 - {}^{-1}\mathfrak{C}x^3 + \ldots \pm {}^{-1}\mathfrak{N}x^n \mp \ldots$$

quae series, si per se ipsam multiplicatur, producit:

$$(1-x)^{-2} = \begin{array}{l|l|l|l|l|l} 1 + 1 & x + 1 & x^2 + 1 & x^3 + 1 & x^4 \ldots 1 & x^n \ldots \\ 1 & 1 & 1 & 1 & 1 & \\ & 1 & 1 & 1 & 1 & \\ & & 1 & 1 & 1 & \\ & & & 1 & 1 & \\ & & & & '' & \\ & & & & 1 & \end{array}$$

$$\text{II. } (1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 \ldots (n+1)x^n \ldots = 1 - {}^{-2}\mathfrak{A}x + {}^{-2}\mathfrak{B}x^2 - {}^{-2}\mathfrak{C}x^3 + \ldots \pm {}^{-2}\mathfrak{N}x^n \mp \ldots$$

$$= 1 + {}^{2}\mathfrak{A}x + {}^{3}\mathfrak{A}x^2 + {}^{4}\mathfrak{A}x^3 + \ldots + {}^{n+1}\mathfrak{A}x^n + \ldots$$

Serie-

Seriebus I. et II. multiplicatione iunctis, provenit:

$$(1-x)^{-3} = 1 + \begin{array}{c|} 2 \\ 1 \end{array} x + \begin{array}{c|} 3 \\ 2 \\ 1 \end{array} x^2 + \begin{array}{c|} 4 \\ 3 \\ 2 \\ 1 \end{array} x^3 + \begin{array}{c|} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{array} x^4 \ldots \begin{array}{c|} (n+1) \\ n \\ (n-1) \\ (n-2) \\ (n-3) \\ '' \\ 1 \end{array} x^n \ldots$$

$$\text{III. } (1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + 15x^4 \ldots + \frac{(n+2).(n+1)}{1\ .\ 2} x^n \ldots = 1 - {}^{-3}\mathfrak{A}x + {}^{-3}\mathfrak{B}x^2 - {}^{-3}\mathfrak{C}x^3 + \ldots \pm {}^{-3}\mathfrak{N}x^n \pm \ldots$$
$$= 1 + {}^{3}\mathfrak{B}x + {}^{4}\mathfrak{B}x^2 + {}^{5}\mathfrak{B}x^3 + \ldots + {}^{n+2}\mathfrak{B}x^n + \ldots$$

Ponatur pariter esse:

$$(1-x)^{-k-1} = 1 - {}^{-k-1}\mathfrak{A}x + {}^{-k-1}\mathfrak{B}x^2 - {}^{-k-1}\mathfrak{C}x^3 + \ldots \pm {}^{-k-1}\mathfrak{N}x^n \mp \ldots$$
$$= 1 + {}^{k+1}\mathfrak{K}x + {}^{k+2}\mathfrak{K}x^2 + {}^{k+3}\mathfrak{K}x^3 + \ldots + {}^{k+n}\mathfrak{K}x^n - \ldots$$

et multiplicetur haec series per

$$(1-x)^{-1} = 1 + 1x + 1x^2 + 1x^3 + \ldots\ldots 1x^n + \ldots\ldots$$

productum erit:

$$(1-x)^{-k-2} = 1 + \begin{array}{c|} {}^{k+1}\mathfrak{K} \\ 1 \end{array} x + \begin{array}{c|} {}^{k+2}\mathfrak{K} \\ {}^{k+1}\mathfrak{K} \\ 1 \end{array} x^2 + \begin{array}{c|} {}^{k+3}\mathfrak{K} \\ {}^{k+2}\mathfrak{K} \\ {}^{k+1}\mathfrak{K} \\ 1 \end{array} x^3 + \ldots + \begin{array}{c|} {}^{k+n}\mathfrak{K} \\ {}^{k+n-1}\mathfrak{K} \\ {}^{k+n-2}\mathfrak{K} \\ {}^{k+n-3}\mathfrak{K} \\ '' \\ {}^{k+1}\mathfrak{K} \\ 1 \end{array} x^n + \ldots$$

In hoc producto Coefficiens termini secundi est:

$$\left.\begin{array}{c} {}^{k+1}\mathfrak{K} \\ 1 \end{array}\right\} = k+2 = {}^{k+2}\overset{+1}{\mathfrak{K}}$$

et quia coefficiens quilibet coefficientem priorem integrum involvit, secundum relationem notam ${}^{n}\overset{-1}{\mathfrak{M}} + {}^{n}\mathfrak{M} = {}^{n+1}\mathfrak{M}$ reperitur:

Coefficiens termini tertii $\left.\begin{array}{c} {}^{k+2}\mathfrak{K} \\ {}^{k+1}\mathfrak{K} \\ 1 \end{array}\right\} = \left.\begin{array}{c} {}^{k+2}\mathfrak{K} \\ {}^{k+2}\overset{+1}{\mathfrak{K}} \end{array}\right\} = {}^{k+3}\overset{+1}{\mathfrak{K}},$

Coefficiens termini quarti $\left.\begin{array}{c} {}^{k+3}\mathfrak{K} \\ {}^{k+2}\mathfrak{K} \\ {}^{k+1}\mathfrak{K} \\ 1 \end{array}\right\} = \left.\begin{array}{c} {}^{k+3}\mathfrak{K} \\ {}^{k+3}\overset{+1}{\mathfrak{K}} \end{array}\right\} = {}^{k+4}\overset{+1}{\mathfrak{K}},$

Coeffi-

Coefficiens termini quinti $\left.\begin{array}{r} {}^{k+4}\mathfrak{R} \\ {}^{k+3}\mathfrak{R} \\ {}^{k+2}\mathfrak{R} \\ {}^{k+1}\mathfrak{R} \\ 1 \end{array}\right\} = \left.\begin{array}{r} {}^{k+4}\mathfrak{R} \\ {}^{k+4}\overset{+1}{\mathfrak{R}} \end{array}\right\} = {}^{k+5}\overset{+1}{\mathfrak{R}},$

Coefficiens termini $(n+1)$ $\left.\begin{array}{r} {}^{k+n}\mathfrak{R} \\ {}^{k+n-1}\mathfrak{R} \\ {}^{k+n-2}\mathfrak{R} \\ '' \\ {}^{k+1}\mathfrak{R} \\ 1 \end{array}\right\} = \left.\begin{array}{r} {}^{k+n}\mathfrak{R} \\ {}^{k+n}\overset{+1}{\mathfrak{R}} \end{array}\right\} = {}^{k+n+1}\overset{+1}{\mathfrak{R}},$

quibus patet, esse

$$(1-x)^{-k-2} = 1 + {}^{k+2}\overset{+1}{\mathfrak{R}}x + {}^{k+3}\overset{+1}{\mathfrak{R}}x^2 + {}^{k+4}\overset{+1}{\mathfrak{R}}x^3 \ldots\ldots + {}^{k+n+1}\overset{+1}{\mathfrak{R}}x^n \ldots\ldots,$$

sive propter relationem ${}^{q+p}\mathfrak{Q} = {}^{q+p}\mathfrak{P} = \pm\, {}^{-q-1}\mathfrak{P}$, ubi est $q > p$, *)

$$(1-x)^{-k-2} = 1 - {}^{-k-2}\mathfrak{A}x + {}^{-k-2}\mathfrak{B}x^2 - {}^{-k-2}\mathfrak{C}x^3 \ldots\ldots \pm {}^{-k-2}\mathfrak{N}x^n \mp \ldots\ldots$$

quae demonstratio nititur quidem in assumtione,

$$(1-x)^{-k-1} = 1 - {}^{-k-1}\mathfrak{A}x + {}^{-k-1}\mathfrak{B}x^2 \ldots\ldots$$

sed quia hanc veram esse vidimus pro $k = 1$ et $k = 2$, ipsam demonstrationem admitti oportet.

§. 10.

Monuimus supra in Scholio ad §. 8. in aequatione:

$$(1-x)^u = 1 - {}^u\mathfrak{A}x + {}^u\mathfrak{B}x^2 - {}^u\mathfrak{C}x^3 + {}^u\mathfrak{D}x^4 - \ldots\ldots$$

valores eosdem posse tribui exponenti u, qui ei in aequatione log. nat. $(1-x)^u$ $= $ log. nat. $(1-x)$ dari possint. Iam videamus, utrum haec aequatio logarithmorum omnes valores exponentis u admittat, an aliquos excludat.

I. Con-

*) Posuimus $q > p$, ergo ${}^{q+p}\mathfrak{Q} = \frac{(q+p)\cdot(q+p-1)\ldots(q+p-p+1)}{1\;.\;2\;\ldots\ldots\;p} \cdot \frac{q\,(q-1)\ldots(p+1)}{(p+1)\cdot(p+2)\ldots q} = \frac{(q+p)\cdot(q+p-1)\ldots(q+1)}{1\;.\;2\;\ldots\;p}$, ${}^{q+q}\mathfrak{P} = \frac{(q+p)\cdot(q+p-1)\ldots(q+1)}{1\;.\;2\;\ldots\;p}$ et $\pm\, {}^{-q-1}\mathfrak{P} = \frac{\pm(-q-1)\cdot(-q-2)\cdot(-q-3)\ldots(-q-p)}{1\;.\;2\;.\;3\;\ldots\;p} = + \frac{(q+p)\cdot(q+p-1)\ldots(q+1)}{1\;.\;2\;\ldots\;p}$, ubi superiora, vel inferiora signa valent, pro paritate, vel imparitate numeri p, ergo ${}^{q+p}\mathfrak{Q} = {}^{q+p}\mathfrak{P} = \pm\, {}^{-q-1}\mathfrak{P}$.

I. Conſtat, Logarithmos naturales eſſe exponentes numeri $h=2,718\ 281\ 8284\ldots$, quem baſin ſyſtematis horum logarithmorum vocant, eamque ob rem eſſe

$$1-x = h^{\log.\,nat.\,(1-x)}$$

$$(1-x)^{\frac{n}{m}} = [h^{\log.\,nat.\,(1-x)}]^{\frac{n}{m}} = h^{\frac{n}{m}\log.\,nat.\,(1-x)}$$

$$\text{et } \log.(1-x)^{\frac{n}{m}} = \frac{n}{m}\log.\,nat.\,(1-x),$$

ubi $\frac{n}{m}=u$ deſignat numerum quemlibet rationalem poſitivum, ſive negativum, integrum, ſive fractum. Sed exponenti u non modo rationales, ſed etiam irrationales et imaginarios valores tribui poſſe et analogia ſuadet et ſequentia affirmabunt.

II. Quia numerus irrationalis quilibet per ſeriem infinitam exprimi poteſt, praebeat nobis exemplum ſeries:

$$\text{Coſ. } n = 1 - \frac{n^2}{1.2} + \frac{n^4}{1.2.3.4} - \frac{n^6}{1.2.3.4.5.6} + \ldots.$$

cuius valor ſolet eſſe irrationalis, ſicuti aliarum ſerierum trigonometricarum.

$$\text{Eſt } (1-x)^{\text{Coſ.}\,n} = (1-x)^{1-\frac{n^2}{1.2}+\frac{n^4}{1.2.3.4}-\frac{n^6}{1.2.3.4.5.6}+\ldots} = (1-x)^1 (1-x)^{\frac{-n^2}{1.2}} (1-x)^{\frac{+n^4}{1.2.3.4}} (1-x)^{\frac{-n^6}{1.2.3.4.5.6}}\ldots$$

$$\text{ergo } \log.(1-x)^{\text{Coſ.}\,n} = \log.(1-x)^1 + \log.(1-x)^{\frac{-n^2}{1.2}} + \log.(1-x)^{\frac{+n^4}{1.2.3.4}}\ldots.$$

$$= [1 - \frac{n^2}{1.2} + \frac{n^4}{1.2.3.4} - \frac{n^6}{1.2.3.4.5.6} + \ldots] \log.(1-x$$

$$= \text{Coſ. } n \ \log.(1-x),$$

quo exemplo, cum ſimilia ratiocinia ad numeros irrationales omnes pertineant, exponentis u irrationalem eſſe poſſe valorem, oſtenditur.

III. Logarithmorum ratio poſcit, ut $\log.\,n.\,y^{\sqrt{-1}}$ factorem habeat $= \log.\,n.\,y$, nullus enim, ſi hoc neges, erit nexus inter logarithmos huius aliarumque potentiarum quantitatis y, praeterea eundem logarithmum factorem imaginarium complecti oportet, niſi numero reali et imaginario idem competat logarithmus. Itaque licebit poni

$$\log.\,y^{\sqrt{-1}} = (a+b\sqrt{-1})\log.\,y,$$

ubi a et b ſignificant quantitates reales, et $(a+b\sqrt{-1})$ cuiuſcunque factoris imaginarii vicem ſuſtinet, quod ad hanc formam numerum imaginarium quemlibet poſſe revocari, Tempelhofius (*Anfangsgründe der Analyſis endlicher*

cher Grösſen 1769. §. 650.) aliique oſtenderunt, de qua re videatur KAESTNERVS *(Anfangsgründe der Analyſis endlicher Gröſsen 1794. pag. 171.)*.

Hinc ſequitur, ut ſit:

$$y^{\sqrt{-1}} = h^{\text{log. nat. } y^{\sqrt{-1}}}$$
$$= h^{(a+b\sqrt{-1})\text{ log. nat. } y}$$
$$= h^{(\text{log. nat. } y).(a+b\sqrt{-1})} \quad \text{et quia } y = h^{\text{log. nat. } y}$$
$$y^{\sqrt{-1}} = y^{a+b\sqrt{-1}},$$

ubi, exponentibus literae y inter ſe comparatis, deprehendimus $\sqrt{-1} = a+b\sqrt{-1}$, ergo $(1-b)\sqrt{-1} = a$, et $a = 0$, quia numerus *realis*, qualem a poſuimus, nullus aequare poteſt imaginarium, et ex eadem hypotheſi a *imaginarius* eſſe numerus non debet. Itaque eſt $(1-b)\sqrt{-1} = 0$ et $1-b = 0$ et $b = 1$. Denique, valoribus literarum a, b, ſubſtitutis, invenitur log. nat. $y^{\sqrt{-1}} = (a+b\sqrt{-1})$; log. nat. $y = \sqrt{-1}$ log. nat. y, et quia omnis imaginaria quantitas ad formam $m+n\sqrt{-1}$ redit, log. nat. $y^{m+n\sqrt{-1}}$ = log. nat. y^m + log. nat. $y^{n\sqrt{-1}} = (m+n\sqrt{-1})$ log. nat. y, quibus, poſito $y = (1-x)$, patet, exponentem u in aequatione

$$\text{log. nat. } (1-x)^u = u \text{ log. nat. } 1-x$$

vel imaginarium habere poſſe valorem.